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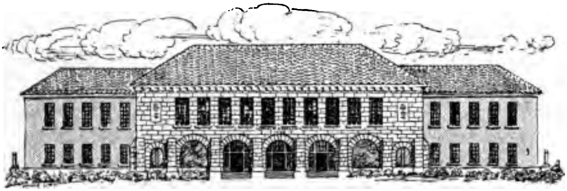

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A
COMPLETE
ALGEBRA

FOR

HIGH SCHOOLS, ACADEMIES, AND NORMAL SCHOOLS.

BY

GEO. W. HULL, M.A., PH.D.,
PROFESSOR OF MATHEMATICS IN THE FIRST PENNSYLVANIA STATE NORMAL SCHOOL, MILLERSVILLE, PA.; AUTHOR OF HULL'S ARITHMETICS.

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PREFACE.

THE object of this work is to give a complete course in all those portions of Algebra which are required in our best High Schools, Academies, and Normal Schools, and at the same time to meet the requirements of students preparing for admission to college.

In the preparation of this work the aim has been to make the transition from Arithmetic to Algebra a natural and easy process; to illustrate and discuss each subject with clearness and sufficient fullness; and to so grade the exercises that the beginner will take up each new topic with increased pleasure and profit, and feel that he is both gaining power and mastering the subject.

Great care has been taken in the selection of examples—neither to make them too difficult, and thus discourage the pupil, nor too easy, and thus deprive him of the power that comes from patient effort. All such problems as merely consume time and do not develop power have been carefully omitted, and yet a sufficient number of well-graded examples will be found under each subject to fix it permanently in the mind of the student.

Attention is called to the simple, clear, and concise method of presenting the subjects of Factoring, Highest Common Divisor, Binomial Theorem, Cube Root, Quadratics, etc.

The Author returns his sincere thanks for the assistance which he has received from experienced teachers.

GEO. W. HULL.

MILLERSVILLE, PA.,
Oct. 1, 1895.

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ALGEBRA.

I. INTRODUCTION.

1. THE object of the following exercises is to lead the pupil by a natural and easy process from the definitions and operations of **Arithmetic** to those of **Algebra.**

EXERCISES.

1. A has a certain sum of money, B has three times as much, and C four times as much as A; how many times A's money have B and C together?

2. If in the above problem we let x represent A's money, what will represent B's money? C's money? B's and C's together?

Three times x is written $3x$, and is read *three x;* four times x is written $4x$, and is read *four x*; a times x is written ax, and is read a . . . . x.

3. What is the sum of $4x$ and $6x$?

4. What is the sum of $2a$, $3a$, and $4a$?

5. What is the sum of y, $4y$, $6y$, and $8y$?

2. The number or letter prefixed to a quantity, showing how many times it is taken, is called the **Coefficient.**

Thus, in $4x$, 4 is the coefficient of x; and in ax, a is the coefficient of x. When no coefficient is expressed, 1 is understood; thus, $x = 1x$.

3. The **Symbols of Operation** used in Arithmetic to indicate addition, subtraction, multiplication, division, and equality are also employed in Algebra.

1. What is the sum of $3x + 4x + 5x$?

2. What is the sum of $2y + 3y + 4y$?

3. What is the sum of $5z + 3z + 4z + 2z$?

4. What is the sum of $5a + 8a + 7a + a$?

5. A has x dollars, B has six times as much, and C twice as much as A; what will represent B's money? C's money? The difference between B's and C's money?

6. What is the value of $8x - 6x$? $10a - 4a$?

7. What is the value of $12m - 5m$? $18n - 10n$?

8. What is the value of $8x - 5x$ when $x = 10$?

9. What is the value of $5y + 3y - 2y$ when $y = 7$?

10. What is the value of $8z - 4z + 5z$ when $z = 10$?

4. The product of two or more quantities expressed by letters is indicated by writing the letters side by side without the multiplication sign between them.

Thus, $a \times x = ax$, and $a \times b \times x = abx$.

1. A has x dollars, and B has a times as much; how much money has B?

2. What is the product of $x \times y$? Of $m \times n$?

3. What is the product of $m \times n \times p$? Of $a \times b \times c$?

4. What is the value of $m \times n$ when $m = 2$ and $n = 4$?

5. What is the value of $a \times b \times c$ when $a = 3$, $b = 4$, and $c = 5$?

6. What is the value of xyz when $x = 2$, $y = 3$, and $z = 4$?

5. The quantities multiplied together are called **Factors** of the product.

Thus, a and b are the factors of ab.

6. The result obtained by using a quantity two or more times as a factor is a **Power** of that quantity. When the quantity is used twice as a factor the result is called the **Square** of the quantity; when used three times, the **Cube**; when used four times, the **Fourth Power**, etc.

The **Exponent** is a number or letter, written slightly above the quantity on the right, to indicate how often the quantity is used as a factor.

Thus, $a \times a = a^2$, read *a square;* $a \times a \times a = a^3$, read *a cube;* $a \times a \times a \times a = a^4$, read *a fourth*, etc.

1. What is the product of $a \times a \times a \times a \times a$?
2. What is the value of a^2 when $a = 6$?
3. What is the value of x^3 when $x = 8$?
4. What is the value of x^2y^2 when $x = 3$ and $y = 2$?
5. What is the value of a^2b^3 when $a = 4$ and $b = 3$?

7. Division may be expressed in several ways.

Thus, x divided by a is written $x \div a$, or $\frac{x}{a}$.

1. A has x dollars, which is a times B's money; what is B's money?
2. What is the value of $x \div y$ when $x = 50$ and $y = 10$?
3. What is the value of $ab \div c$ when $a = 8$, $b = 4$, and $c = 16$?
4. What is the value of $x^3 \div y$ when $x = 10$ and $y = 25$?
5. What is the value of $a^2b^2 \div x$ when $a = 3$, $b = 6$, and $x = 18$?
6. What is one of the two equal factors of 4? 16? 25? x^2? y^2? a^2?
7. What is one of the three equal factors of 8? 27? x^3? y^3? a^3?

8. One of the equal factors of a quantity is called a **Root** of the quantity; one of two equal factors is the **Square Root**; one of three equal factors is the **Cube Root**, etc.

The symbol of evolution is $\sqrt{}$, called the **Radical Sign.** A number or letter, called the **Index**, is written in the angle of the radical sign to show what root is required. When no index is used the square root is understood.

Thus, $\sqrt{4}$ indicates the square root of 4.
$\sqrt[3]{x}$ indicates the cube root of x.
$\sqrt[4]{y}$ indicates the fourth root of y.
$\sqrt[n]{x}$ indicates the nth root of x.

1. What is the value of $\sqrt{144}$? $\sqrt[3]{216}$?
2. What is the value of $\sqrt[3]{a^3b^6}$? $\sqrt{4a^2x^2}$?
3. What is the value of $\sqrt{x}$ when $x = 64$? when $x = 100$?
4. What is the value of $\sqrt[3]{ab}$ when $a = 32$ and $b = 2$?

5. What is the value of $\sqrt{mn}$ when $m = 9$ and $n = 4$?

6. What is the value of $\sqrt[3]{xy}$ when $x = 5$ and $y = 25$?

SOLUTION OF PROBLEMS BY ALGEBRA.

9. Illustration.—B has five times as much money as A, and they together have $60; how much has each?

Arithmetical Process.

A's money + *five times* his money = $60.
Hence six times A's money = $60.
And A's money = one sixth of $60, or $10.
And B has five times $10, or $50.

Algebraic Process.

Let x = A's money.
And $5x$ = B's money.
Then $6x$ = A's and B's money.
Hence $6x$ = $60.
And x = one sixth of $60 = $10, A's money.
$5x$ = five times $10 = $50, B's money.

10. An **Equation** is an expression of equality between two quantities.

Thus, $2 \times 4 = 8$, $8 + 5 = 13$, $6x = 60$ are equations.

EXAMPLES AND PROBLEMS.

Solve the following equations and problems:

1. $8x = 40$.
2. $10x = 100$.
3. $4x + 3x = 84$.
4. $8x - 5x = 120$.
5. $9x + 2x = 121$.
6. $9y + 3y - 8y = 400$.
7. $6z - 2z + 8z = 144$.
8. $12x + 16x - 8x = 220$.
9. $8y + 10y - 4y = 280$.
10. $x + 11x + 6x = 360$.

11. The sum of two numbers is 132, and the greater is ten times the less; what are the numbers?

12. A is six times as old as B, and the sum of their ages is 56 years; what is the age of each?

13. M and N together own 850 acres of land, and M owns nine times as much as N; how many acres has each?

14. A, B, and C together have $8,100: B has twice as much as A, and C has three times as much as B; what is the fortune of each?

15. Divide $36,100 among R, S, and T, so that T shall have five times as much as S, and S three times as much as R.

16. The difference between two numbers is 120, and the larger is five times the smaller; what are the numbers?

PROCESS.

Let $x =$ the smaller number.

Then $5x =$ the larger number.

And $5x - x = 120$.

Or $4x = 120$.

$x = 30$, the smaller number.

And $5x = 150$, the larger number.

17. The difference between two numbers is 150, and eleven times the smaller number equals the larger; what are the numbers?

18. Ten times a number diminished by three times the number equals 63; what is the number?

19. If five times a number be taken from twelve times the number, the result will be 98; what is the number?

20. A horse cost $120 more than a wagon; what was the cost of each if the horse cost five times as much as the wagon?

21. A's house cost eight times as much as his barn; what was the cost of each if the house cost $6,300 more than the barn?

22. The greater of two numbers is four times the less, and their sum is 400; what are the numbers?

23. A house and lot cost \$1,900; what was the cost of each if the house cost \$900 more than the lot?

PROCESS.

Let $x =$ the cost of the lot.
Then $x + \$900 =$ the cost of the house.
And $2x + \$900 =$ the cost of both.
Hence $2x + \$900 = \$1,900.$
And $2x = \$1,900 - \$900 = \$1,000.$
$x = \$500,$ the cost of the lot.
And $x + \$900 = \$1,400,$ the cost of the house.

24. A has \$600 more than B, and they together have \$2400; how much money has each?

25. A boy being asked how much money he had, replied, "Five times my money increased by \$20 is \$100;" how much money had he?

26. C's horse and carriage cost \$350; what was the cost of each if the horse cost twice as much as the carriage, plus \$50?

27. A and B enter into partnership with a joint capital of \$3,500; what is the share of each if A owns seven times as much as B, plus \$300?

28. Divide the number 810 into three such parts that the second may be 50 more than the first, and the third 50 more than the second.

29. If a number is multiplied by 8, and this product diminished by 40, the result is 320; what is the number?

PROCESS.

Let $x =$ the number.
Then $8x - 40 = 320.$
$8x = 320 + 40 = 360.$
$x = 45.$

30. A's house and barn cost \$8,740; what was the cost of each if the house cost three times as much as the barn, minus \$160?

31. Divide the number 640 into three such parts that the second may be 60 less than the first, and the third 100 more than the second.

32. $8x + 5x - 3x + 90 = 240.$

33. $9x - 7x + 12x - 75 = 205.$

34. $10y + 8y - 9y - 63 = 72.$

35. $11x - 5x + 18 - x = 48.$

36. $14x + 6x - 5x + 12 = 117.$

37. $20y - 8y - 4y - 60 = 420.$

38. A horse and carriage cost \$420; what was the cost of each if the carriage cost two thirds as much as the horse?

PROCESS.

Let $x =$ the cost of the horse.

Then $\frac{2}{3}x =$ the cost of the carriage.

And $x + \frac{2}{3}x = \$420.$

Or $\frac{5}{3}x = \$420.$

And $\frac{1}{3}x = \$84.$

$x = \$252$, the cost of the horse.

$\frac{2}{3}x = \$168$, the cost of the carriage.

39. What number is that to which if its four fifths be added the sum will be 117?

40. What number is that which being diminished by its two fifths equals 80?

41. If from five times a number you subtract two thirds of the number, the remainder is 260; what is the number?

42. Five times the distance from Philadelphia to New York diminished by $2\frac{1}{3}$ times the distance equals 240 miles; required the distance.

43. A farmer bought a horse and a cow for \$360, paying five times as much for the horse as for the cow; required the cost of each.

44. A pole, 120 ft. high, in falling broke into three unequal parts: the longest piece was three times the length of the shortest, and the length of the other piece was two times the length of the shortest; required the length of each piece.

45. What number added to six times itself, and 10 more, equals 108?

46. Gilbert has five times as much money as Harold, and Harold has three times as much as Harry; how much has each if Gilbert has $210 more than Harry?

47. If four times a number increased by 17 equals 81, what is that number?

48. If M's age be increased by its three fourths and 9 years more, it will equal 51 years; what is his age?

49. Four fifths of a sum of money, increased by three fourths of the same sum and $8 more, equals $101; what is the sum of money?

50. A watch and chain cost $120; what was the cost of each if the chain cost two fifths as much as the watch plus $5?

51. Six times a certain number, plus five times the sum obtained by increasing the number by 5, equals 245; what is the number?

52. A had twice as much money as B, and B had three times as much as C; if they all had $200, how much had each?

53. A farmer has 108 animals, consisting of horses, sheep, and cows: he has four times as many cows as horses lacking 8, and five times as many sheep as horses lacking 4; how many has he of each kind?

54. A, B, and C enter into partnership with a joint capital of $26,250: B furnishes $2\frac{1}{2}$ times as much as A, and C furnishes one half as much as A and B together; how much did each furnish?

55. A man built a barn, a house, and a store for $65,000: the house cost three times as much as the barn, and the store cost three times as much as the house; what was the cost of each?

II. DEFINITIONS AND NOTATION.

11. **Mathematics** is the science of quantity.

12. **Quantity** is the amount or extent of that which may be measured. It is of two kinds, Number and Extension.

13. **Number** is quantity conceived as made up of equal parts, and answers the question, "How many?"

14. **Extension** is quantity conceived as made up of equal parts, and answers the question, "How much?"

15. **Arithmetic** is the science of *number*. **Geometry** is the science of *extension*.

16. **Algebra** treats of the investigation of *quantity* by means of general characters, called *symbols*.

17. The **Symbols** of Algebra are of four kinds:

1. Symbols of Quantity.
2. Symbols of Operation.
3. Symbols of Relation.
4. Symbols of Abbreviation.

SYMBOLS OF QUANTITY.

18. The **Symbols** generally used to represent quantity are the figures of Arithmetic and the letters of the alphabet.

19. **Known Quantities** are those whose values are given, and are represented by figures, 1, 2, 3, etc., or by the *first* letters of the alphabet, as a, b, c, etc.

20. **Unknown Quantities** are those whose values are not given, and are represented by the *final* letters of the alphabet, as x, y, z, etc.

21. **Zero,** 0, denotes the absence of quantity, or that which is less than any assignable quantity.

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43. A farmer bought a horse and a cow for $360, paying five times as much for the horse as for the cow; required the cost of each.

42. **Similar Terms** are those containing the same letters affected by the same exponents. The coefficients and signs may differ.

Thus, $4ab$ and $-6ab$ are similar terms; also, $-3a^2b^3$ and $5a^2b^3$ are similar terms.

43. **Dissimilar Terms** are those containing different letters or different exponents.

Thus, ab and cd are dissimilar terms; also, ax^2 and a^2x are dissimilar terms.

44. The **Degree** of a term is determined by the number of literal factors it contains.

Thus, $3a$ is of the *first* degree; $4a^2$ or $-5ab$, of the *second* degree $2abc$, a^2b, and c^3 are of the *third* degree, etc.

45. **Homogeneous Terms** are those which are of the same degree.

Thus, $4xyz$ and $-2x^2y$ are homogeneous terms.

46. A **Monomial** is an algebraic expression consisting of one term; as, $3ab$ or $-5ax$.

47. A **Polynomial** is an algebraic expression consisting of more than one term; as, $x+y$, $x^2+2xy+y^2$, or $x^3+3x^2y+3xy^2+y^3$.

48. A **Binomial** is a polynomial of two terms; as, $x+y$.

49. A **Trinomial** is a polynomial of three terms; as, $m^2+2mn+n^2$.

50. The **Reciprocal** of a quantity is 1 divided by that quantity.

Thus, the reciprocal of x is $\frac{1}{x}\cdot$

51. The **Absolute Value** of a quantity is its value taken independent of its sign.

Thus, $+5a$ and $-5a$ have the same absolute value.

52. The **Numerical Value** of an algebraic expression is

II. DEFINITIONS AND NOTATION.

11. Mathematics is the science of quantity.

12. Quantity is the amount or extent of that which may be measured. It is of two kinds, Number and Extension.

13. Number is quantity conceived as made up of equal parts, and answers the question, "How many?"

14. Extension is quantity conceived as made up of equal parts, and answers the question, "How much?"

15. Arithmetic is the science of *number*. **Geometry** is the science of *extension*.

16. Algebra treats of the investigation of *quantity* by means of general characters, called *symbols*.

17. The **Symbols** of Algebra are of four kinds:

1. Symbols of Quantity.
2. Symbols of Operation.
3. Symbols of Relation.
4. Symbols of Abbreviation.

SYMBOLS OF QUANTITY.

18. The **Symbols** generally used to represent quantity are the figures of Arithmetic and the letters of the alphabet.

19. Known Quantities are those whose values are given, and are represented by figures, 1, 2, 3, etc., or by the *first* letters of the alphabet, as a, b, c, etc.

20. Unknown Quantities are those whose values are not given, and are represented by the *final* letters of the alphabet, as x, y, z, etc.

21. Zero, 0, denotes the absence of quantity, or that which is less than any assignable quantity.

33. Show that $(x-y)^2=x^2-2xy+y^2$.

34. Show that $(x+y)^3=x^3+3x^2y+3xy^2+y^3$.

35. Show that $(x-y)^3=x^3-3x^2y+3xy^2-y^3$.

36. Show that $\dfrac{x^6-1}{x^3+1}=x^3-1$.

37. Show that $\dfrac{x^6-1}{x^2-1}=x^4+x^2+1$.

38. Show that $\dfrac{x^6+1}{x^2+1}=x^4-x^2+1$.

39. Show that $\dfrac{x^5-1}{x-1}=x^4+x^3+x^2+x+1$.

40. Show that $\dfrac{x^5+1}{x+1}=x^4-x^3+x^2-x+1$.

AXIOMS.

54. An **Axiom** is a self-evident truth. The following axioms, together with the preliminary definitions, form the basis of algebraic reasoning:

1. If equals be added to equals, the sums will be equal.
2. If equals be subtracted from equals, the remainders will be equal.
3. If equals be multiplied by equals, the products will be equal.
4. If equals be divided by equals, the quotients will be equal.
5. If the same quantity be both added to and subtracted from another, the value of the latter will not be changed.
6. If a quantity be both multiplied and divided by the same quantity, the former will not be changed.
7. Like powers of equal quantities are equal.
8. Like roots of equal quantities are equal.
9. Quantities that are equal to the same quantity are equal to each other.
10. The whole is equal to the sum of all its parts.

II. DEFINITIONS AND NOTATION.

11. **Mathematics** is the science of quantity.

12. **Quantity** is the amount or extent of that which may be measured. It is of two kinds, Number and Extension.

13. **Number** is quantity conceived as made up of equal parts, and answers the question, "How many?"

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15. **Arithmetic** is the science of *number*. **Geometry** is the science of *extension*.

16. **Algebra** treats of the investigation of *quantity* by means of general characters, called *symbols*.

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19. **Known Quantities** are those whose values are given, and are represented by figures, 1, 2, 3, etc., or by the *first* letters of the alphabet, as a, b, c, etc.

20. **Unknown Quantities** are those whose values are not given, and are represented by the *final* letters of the alphabet, as x, y, z, etc.

21. **Zero,** 0, denotes the absence of quantity, or that which is less than any assignable quantity.

22. The greater of two numbers is four times the less, and their sum is 400; what are the numbers?

23. A house and lot cost \$1,900; what was the cost of each if the house cost \$900 more than the lot?

PROCESS.

Let $x =$ the cost of the lot.

Then $x + \$900 =$ the cost of the house.

And $2x + \$900 =$ the cost of both.

Hence $2x + \$900 = \$1,900$.

And $2x = \$1,900 - \$900 = \$1,000$.

$x = \$500$, the cost of the lot.

And $x + \$900 = \$1,400$, the cost of the house.

24. A has \$600 more than B, and they together have \$2400; how much money has each?

25. A boy being asked how much money he had, replied, " Five times my money increased by \$20 is \$100;" how much money had he?

26. C's horse and carriage cost \$350; what was the cost of each if the horse cost twice as much as the carriage, plus \$50?

27. A and B enter into partnership with a joint capital of \$3,500; what is the share of each if A owns seven times as much as B, plus \$300?

28. Divide the number 810 into three such parts that the second may be 50 more than the first, and the third 50 more than the second.

29. If a number is multiplied by 8, and this product diminished by 40, the result is 320; what is the number?

PROCESS.

Let $x =$ the number.

Then $8x - 40 = 320$.

$8x = 320 + 40 = 360$.

$x = 45$.

30. A's house and barn cost \$8,740; what was the cost of each if the house cost three times as much as the barn, minus \$160?

31. Divide the number 640 into three such parts that the second may be 60 less than the first, and the third 100 more than the second.

32. $8x + 5x - 3x + 90 = 240.$

33. $9x - 7x + 12x - 75 = 205.$

34. $10y + 8y - 9y - 63 = 72.$

35. $11x - 5x + 18 - x = 48.$

36. $14x + 6x - 5x + 12 = 117.$

37. $20y - 8y - 4y - 60 = 420.$

38. A horse and carriage cost \$420; what was the cost of each if the carriage cost two thirds as much as the horse?

PROCESS.

Let $x =$ the cost of the horse.

Then $\frac{2}{3}x =$ the cost of the carriage.

And $x + \frac{2}{3}x = \$420.$

Or $\frac{5}{3}x = \$420.$

And $\frac{1}{3}x = \$84.$

$x = \$252$, the cost of the horse.

$\frac{2}{3}x = \$168$, the cost of the carriage.

39. What number is that to which if its four fifths be added the sum will be 117?

40. What number is that which being diminished by its two fifths equals 80?

41. If from five times a number you subtract two thirds of the number, the remainder is 260; what is the number?

42. Five times the distance from Philadelphia to New York diminished by $2\frac{1}{3}$ times the distance equals 240 miles; required the distance.

43. A farmer bought a horse and a cow for \$360, paying five times as much for the horse as for the cow; required the cost of each.

2. Find the sum of $2a - 3b$, $5b - 6c$, $4a - 5c$, $-a + 8b$, and $2a - b + c$.

PROCESS.

$$\begin{array}{rrr} 2a & -3b & \\ & 5b & -6c \\ 4a & & -5c \\ -a & +8b & \\ 2a & -b & +c \\ \hline 7a & +9b & -10c, \text{ Ans.} \end{array}$$

For convenience in adding we write the quantities so that similar terms are in the same column.

The columns are added as in ART. 56. The sum of the first column is $7a$, of the second column is $9b$, and of the third column is $-10c$. Hence the sum is $7a + 9b - 10c$.

From these examples we derive the following

RULE.

Write the quantities so that similar terms shall be in the same vertical column. Add each column separately, and connect the results with their proper signs.

EXAMPLES.

3.	4.	5.
$\begin{array}{rr} 7x & -5y \\ -x & +3y \\ 4x & -y \\ -3x & +4y \\ \hline \end{array}$	$\begin{array}{rr} 5ab & -8xy \\ -3ab & +4xy \\ +2ab & -11xy \\ -ab & -2xy \\ \hline \end{array}$	$\begin{array}{rrr} a & -2b & \\ & +5b & -4c \\ 3a & -4b & \\ -a & -3b & +8c \\ \hline \end{array}$

6. Add $3a - 5b$, $4a + 2b$, $a - 3b$, $6a + 4b$, $14a - 2b$, and $4a - 5b$.

7. Add $2x - 5ay$, $11x + 4ay$, $a - 3x + 5ay$, $3a - 4x - 7ay$, and $7x - 4ay + 5a$.

8. Add $2a - 3b - c$, $4a - 5c$, $5b + 8c$, $-6a - 7b + 2c$, and $3a + 2b - 4c$.

9. Add $4x^3 - 5x^2 + 3x$, $6x^3 + 2x^2 - 5x$, $7x^3 + 9x^2 - 11x$, $-4x^3 - 12x^2 + 15x$, and $7x^3 + 20x^2 + 6x$.

10. Add $a^2 - 4bd + c - 7$, $6a^2 + 5bd + 11c + 2$, $-6a^2 - 3 - 4c - 7bd$, $14c - 11 - 8a^2$, and $8bd + 4 + 2a^2 - 7c$.

11. Add $3x + 4y + 7z$, $3z + 2y + x$, $-4x + 7z - 5y$, $-2x - 12z + 3y$, and $8x - 3y + 9z$.

12. Add $2a - 3b + 4c - 5d$, $d + 2b - 3c$, $3a - 6d + c$, $-2a - 2b + 2d$, $a + b + c$, and $-2a - b - c + 3d$.

13. Add $a^2 + b^2 - c^2 - d^2$, $3a^2 - 3b^2 + 3c^2$, $4b^2 - 4d^2 - 4a^2$, $8d^2 - 5c^2 + 5a^2$, $6b^2 - 6c^2 - 6d^2$, and $a^2 - b^2 - c^2 - d^2$.

14. Add $a^3 - 3a^2b + 3ab^2 - b^3$, $a^3 + 3a^2b + 3ab^2 + b^3$, and $a^2 - 2a^3 - 6ab^2$.

15. Add $m^5 - 4m^3n^2 + 6m^2n^3 + 3n^5$, $6m^5 + 6m^3n^2 - 11m^2n^3 - 6n^5$, $9m^5 - 2m^3n^2 + 4m^2n^3 + 5n^5$, and $-16m^5 + 5m^3n^2 + m^2n^3 - 2n^5$.

16. Add $2x^3y^2 - 5x^2y^3 - x^2y - xy^2 + 4xy + 7x^2y^3 - 3x^2y + 2x^3y^2 + 2xy^2 + xy + x^3y^2 + xy - xy^2 - 2x^2y^3 + 4x^2y$.

17. Add $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$, $4x^3y - 12x^2y^2 - 12xy^3 - 4y^4$, $6x^2y^2 - 12xy^3 + 6y^4$, and $4xy^3 - 4y^4$.

18. Add $x^3 - 3x^2y^2 - y^3 - yz + y^2$, $2y^3 + 7x^2y^2 + 3y^2 - 9$, $4yz + 3 + 3x^3 - 5y^3 + 3x^2y^2$, $2y^3 - 6x^2y^2 + 2y$, $-3yz - x^2y^2 + 4y^2$, $6 - 5y^2$, and $-4x^3 + 3y^3 - 3y^2 - 3y$.

19. Add $7\sqrt{y} - 4(a + b)$, $6\sqrt{y} + 2(a + b)$, $2\sqrt{y} + (a + b)$, $\sqrt{y} - 2(a + b)$, $8\sqrt{y} + 12(a + b)$, $9\sqrt{y} - 5(a + b)$, $-11\sqrt{y} - 10(a + b)$, and $\sqrt{y} - (a + b)$.

20. Add $8(a - x) + 9(x + y) - \sqrt{x}$, $4(a - x) - 6(x + y) + 4\sqrt{x}$, $-5(a - x) - 20(x + y) - 7\sqrt{x}$, $9(a - x) + 7(x + y) + 3\sqrt{x}$, $9(a - x) - (x + y) - 5\sqrt{x}$, $7(a - x) - 8(x + y) - 11\sqrt{x}$, and $-10(a - x) - 4(x + y) - \sqrt{x}$.

21. Add $6(a - x) + 5(x - y)^2 - 4(a + x) + 3(x - a) + 14(a + x) - 15(x - y)^2 - 3(a - x) - 4(x - a) - 5(a + x) - 3(x - a) + 20(x - y)^2 - 9(a - x) - 10(x - y)^2 - 14(a + x) + 4(x - a)$.

22. Add $x^2 + 2xy + y^2$, $x^2 - 2xy + y^2$, $x^2 + xy + y^2$, $x^2 - xy + y^2$, $x^2 + 4xy - y^2$, $x^2 - 5xy - y^2$, $-9x^2 - 6xy - 9y^2$, $16x^2 - 8xy + 16y^2$, $6x^2 - 12xy + 6y^2$, $10x^2 + 10xy - 10y^2$, $4x^2 - xy - 2y^2$, and $3x^2 + 2xy + 3y^2$.

23. Add $x + 2\sqrt{xy} + y$, $x - 2\sqrt{xy} + y$, $6x + 7y$, $8\sqrt{xy} - 4y$, $-5x + \sqrt{xy}$, $3x - 3\sqrt{xy} + 3y$, $4x + 8\sqrt{xy} + 4y$, $-x + 10\sqrt{xy} - 3y$, $x + \sqrt{xy}$, $y - \sqrt{xy}$, and $5x + 10\sqrt{xy} + 5y$.

24. Add $\frac{1}{4}x^2 + xy + \frac{1}{4}y^2$, $\frac{1}{4}x^2 - xy + \frac{1}{4}y^2$, $\frac{1}{2}x^2 - 3xy - 2y^2$, $\frac{1}{8}x^2 + 5xy + \frac{1}{2}y^2$, $\frac{3}{4}x^2 - 4xy - \frac{3}{4}y^2$, $\frac{3}{2}x^2 - \frac{1}{2}xy - 3y^2$, and $-5x^2 + 3xy$.

25. Add $4\sqrt{a+b} + bc - a$, $3a - 5\sqrt{a+b} - 4bc$, $8\sqrt{a+b} - bc$, $5a + 10bc$, $4a + 3\sqrt{a+b} - 7bc$, $4a - 6\sqrt{a+b} - bc$, $9\sqrt{a+b} - a + 3bc$, $5\sqrt{a+b} - 8a + 5bc$, and $9bc - 8\sqrt{a+b} - 7a$.

58. **Dissimilar Terms** having a *common factor* may be added by taking the algebraic sum of the dissimilar parts, inclosing it in parentheses, and prefixing it to the common factor.

EXAMPLES.

1. Add ay^3, by^3, and cy^3.

PROCESS.

$$\begin{array}{c} ay^3 \\ by^3 \\ cy^3 \\ \hline (a+b+c)y^3, \textit{ Ans}, \end{array}$$

It is evident that the sum of a times y^3 plus b times y^3 plus c times y^3 is $a + b + c$ times y^3, or $(a + b + c)y^3$.

2. Add $3x^2$, ax^2, $4x^2$, bx^2, $2ax^2$, and $3bx^2$.

3. Add $2ax$, $3bx$, $4cx$, $-5ax$, $-bx$, and cx.

4. Add $2\sqrt{ax}$, $b\sqrt{ax}$, $c\sqrt{ax}$, $-\sqrt{ax}$, and $-2c\sqrt{ax}$.

5. Add $3a\sqrt{x+y}$, $2\sqrt{x+y}$, $b\sqrt{x+y}$, $2a\sqrt{x+y}$, and $-\sqrt{x+y}$.

6. Add $4(a^2 - b^2)$, $a(a^2 - b^2)$, $-5a(a^2 - b^2)$, and $2(a^2 - b^2)$.

7. Add $a\sqrt{x-y}$, $b\sqrt{x-y}$, $3\sqrt{x-y}$, $-2a\sqrt{x-y}$, and $4b\sqrt{x-y}$.

8. Add $3(a + b - 2)$, $a(a + b - 2)$, $b(a + b - 2)$, $-2(a + b - 2)$, $-b(a + b - 2)$, and $(a + b - 2)$.

9. Add $4\sqrt{a-b}$, $b\sqrt{a-b}$, $c\sqrt{a-b}$, $-2\sqrt{a-b}$, $-2b\sqrt{a-b}$, $-3c\sqrt{a-b}$, $3\sqrt{a-b}$, and $c\sqrt{a-b}$.

EQUATIONS AND PROBLEMS.

59. Review the definition of an **Equation,** ART. 10.

$4x + 2x = 12$ is an equation.

The *Solution* of an equation is the process of finding the value of the unknown quantity.

60. A **Problem** is a question to be solved. The *solution* of a problem is the process of finding the required result.

EQUATIONS.

1. Solve the equation $2x + 8x - 3x + 6x = 156$.

SOLUTION.

$$2x + 8x - 3x + 6x = 156.$$

Uniting the terms, $13x = 156$.

Dividing by 13, $x = 12$, *Ans.*

Solve the following equations:

2. $6x + 8x - 5x = 45$.
3. $11x - 5x + 6x = 144$.
4. $10x + 14x - 7x = 170$.
5. $15y - 10y + 18y = 92$.
6. $20y + 40y - 15y = 180$.
7. $5x + 8x = 16 + 10$.
8. $40x - 16x = 70 + 74$.
9. $16y + 8y - 2y = 110$.
10. $110y - 45y = 80 + 180$.
11. $150x + 105x = 295 - 40$.
12. $4x + 8x + 16x - 10x - 20x = 200 - 60$.
13. $40x - 12x + 13x - 10x = 200 + 311 - 170$.
14. $31x - 15x - 10x + 50x - 2x = 102 + 168$.
15. $5x + 20x - 14x + 12x - 16x = 480 - 60$.
16. $22x - 4x + 32x - 10x = 500 - 70 + 10$.
17. $47x - 20x = 9 \times 120$.
18. $500x - 46x + 90x = 1{,}088$.
19. $421y + 516y = 10{,}307$.
20. $125x - 40x - 22x = 252$.
21. $65x - 32x = -15 - 7$.
22. $8x + 31x = 81 - 159$.
23. $12x + 14x - 8x = 540$.
24. $48x - 12x + 20x = 112$.
25. $90y + 60y + 18y = 672$.
26. $110y - 82y - y = 1{,}350$.

PROBLEMS.

Solve the following problems:

27. The sum of two numbers is 560; what are the numbers if the larger is six times the smaller?

SOLUTION.

Let $x =$ the smaller number.

Then $6x =$ the larger number.

And $x + 6x =$ their sum.

By the conditions, $x + 6x = 560$.

And $7x = 560$.

$x = 80$, smaller, *Ans.*

$6x = 480$, larger, *Ans.*

28. Divide 240 into two such parts that the larger may be five times the smaller.

29. Divide the number 120 into three parts, so that the first shall be twice and the second three times the third.

30. Divide the number 250 into three parts, so that the first shall be two times and the second three times the third.

31. A and B together have $10,000; how much has each if B has four times as much as A?

32. Arthur has six times as much money as Horace, and the difference of their fortunes is $2,550; how much has each?

33. A farmer bought a horse, a cow, and a sheep for $264: the cow cost eight times as much as the sheep, and the horse three times as much as the cow; required the cost of each.

34. A bought two farms, paying eleven times as much for the first as for the second; what was the cost of each if both farms cost $8,800?

35. A drover bought 20 sheep, 10 cows, and 5 horses for $1,250; what was the cost of each if a cow cost three times and a horse fifteen times as much as a sheep?

36. A employs 12 boys, 30 women, and 40 men, whose daily wages are $96; what is the daily wages of each if a woman receives twice and a man three times as much as a boy?

SUBTRACTION.

61. Subtraction is the process of finding the difference between two quantities.

The **Minuend** is the quantity from which we subtract.

The **Subtrahend** is the quantity to be subtracted.

The **Remainder** is the result obtained by subtracting.

62. To subtract when the terms are positive.

1. Subtract $5a$ from $9a$.

PROCESS.

$9a$
$\underline{5a}$
$4a$, *Ans.*

It is evident that nine times a quantity minus five times that quantity equals four times that quantity. Hence $9a$ minus $5a$ equals $4a$.

2. Subtract $9a$ from $5a$.

PROCESS.

$5a = 9a - 4a$
$\underline{9a = 9a}$
$-4a \qquad -4a$, *Ans.*

$5a$ equals $9a - 4a$; $9a$ subtracted from $9a - 4a$ leaves $-4a$. Hence $9a$ subtracted from $5a$ leaves $-4a$.

3. Subtract $m + n$ from a.

PROCESS.

a
$\underline{m + n}$
$a - m - n$, *Ans.*

If m is taken from a, the remainder is $a - m$; but we wish to subtract m *increased* by n from a; we have therefore subtracted n less than is required, and the remainder, $a - m$, is n too large. Taking n from $a - m$, the true remainder is $a - m - n$.

From these examples we see that if we change the signs of the subtrahend and then add, we shall obtain the same result as given in the solution. Hence the

RULE.

Conceive the signs of the subtrahend to be changed, and proceed as in addition.

EXAMPLES.

Solve by subtraction—

4.	5.	6.	7.	8.
$8x$	$3x$	$14ax^2$	$5ax^2$	$10x^2 + 4y^2$
$3x$	$8x$	$5ax^2$	$14ax^2$	$4x^2 + 10y^2$

9. From $8mx + 2ny$ take $7mx + 8ny$.
10. From $3a + b + c$ take $a + 2b + 2c$.
11. From $a^2 + b^2$ take $a^2 + 2ab + b^2$.
12. From $8xy + 3z^2$ take $5xy + 5z^2$.
13. From $5m^2 + 4mn$ take $3m^2 + 6mn + n^2$.
14. From $12b + d + a^2$ take $4a^2 + 6b$.
15. From $a^2 + 3ab + b^2$ take $a^2 + 2ab + b^2$.
16. From $9mp^2 + 3nq$ take $4mp^2 + 6nq$.
17. From $6n + m + a$ take $8n + 8a + \frac{1}{2}m$.
18. From $x^4 + x^2y^2 + y^4$ take $x^4 + 2x^2y^2 + y^4$.
19. From $5x^3 + x^2 + x + 1$ take $3x^3 + x^2 + x + 3$.
20. From $9m^2 + 5mn + 2n^2$ take $8m^2 + 5mn + n^2$.
21. From $x^4 + 5x^3 + 4x + 1$ take $x^4 + 4x^3 + 6x + 4$.
22. From $15xy + 9x^2y^3 + 2x$ take $15x^2y^2 + 10xy$.
23. From $3m^2n + 3mn^2 + 2n^3$ take $m^3 + 3m^2n + 3mn^2 + n^3$.
24. From $18a^2x^2 + 12ay + 10y^2z$ take $12a^2x^2 + 9y^2z$.
25. From $80m^2 + 60mn + 70n^2$ take $79m^2 + 62mn + 69n^2$.

63. To subtract when some of the terms are negative.

1. Subtract $-m$ from a.

PROCESS.

$$\begin{array}{rcr} a & = & a + m - m \\ \underline{-m} & = & \underline{\qquad -m} \\ a + m & & a + m, \text{ Ans.} \end{array}$$

a equals $a + m - m$; $-m$ taken from $a + m - m$ leaves $a + m$. Hence $-m$ subtracted from a leaves $a + m$.

2. Subtract $m - n$ from a.

PROCESS.

$$\begin{array}{l} a \\ \underline{\quad m - n} \\ a - m + n, \text{ Ans.} \end{array}$$

If m is taken from a, the remainder is $a - m$; but we wish to subtract m *diminished* by n from a; we have therefore subtracted n more than is required, and the remainder, $a - m$, is n too small. Adding n to $a - m$, the true remainder is $a - m + n$.

From these examples we again see that if we conceive the signs of the subtrahend changed, and then add, we shall obtain the same result as given in the solution. Hence the

RULE.

Conceive the signs of the subtrahend to be changed, and proceed as in addition.

EXAMPLES.

	3.	4.	5.	6.	7.
From	a	a	$-a$	$-a$	$14a$
Take	c	$-c$	c	$-c$	$-9a$

	8.	9.	10.	11.	12.
From	$9a$	$-14a$	$-9a$	$-14a$	$-9a$
Take	$-14a$	$9a$	$14a$	$-9a$	$-14a$

	13.	14.	15.
From	$a^2 + 2ab + b^2$	$a - b + c$	$a^2 \qquad + b^2$
Take	$a^2 - 2ab + b^2$	$a + b - c$	$a^2 - 2ab + b^2$

16. From $x^2 + 2xy + y^2$ take $x^2 - 2xy - y^2$

17. From $a + b + c$ take $2b + 2c + d$.

18. From $x^2 + y^2$ take $x^2 - 2xy - y^2$.

19. From $4x^4 + 4y^4$ take $4x^4 - 2x^2y^2 + 4y^4$.

20. From $a^m + y^m$ take $a^m - ny + y^m$.

21. From $a^2 + a + 1$ take $a^2 - 3a + 2$.

22. From $\frac{1}{2}a^2 + \frac{3}{4}b^2$ take $\frac{1}{6}a^2 - \frac{2}{3}b^2 - 4$.

23. From $2x - \frac{1}{3}y + 6$ take $\frac{1}{2}x - y - 8$.

24. From $x^{2m} + x^my^n + y^{2n}$ take $x^{2m} - 3x^my^n - y^{2n}$.

25. From $a^3 + 3a^2b + 3ab^2 + b^3$ take $a^3 - 3a^2b + 3ab^2 - b^3$.

26. From the sum of $a^2 + 2ab - b^2$ and $3ab - 2b^2 + c$ take $a^2 + 5ab - 3b^2 - c$.

27. From the sum of $x^{2n} - 4x^ny^n + y^{2n}$ and $x^{2n} + 4x^ny^n + y^{2n}$ take $4x^ny^n - 3y^{2n}$.

28. From $x^8 - 4x^4y^4 - y^8$ minus $- 8x^4y^4 - 2y^8$ take $4x^4y^4 - 16$.

29. From the sum of $a^3 + 3a^2b + 3ab^2 + b^3$ and $a^3 - 3a^2b + 3ab^2 - b^3$ take their difference.

30. From the sum of $m^2 + 3mn - n^2$ and $m^2 - 4mn + n^2$ take $m^2 + 5mn - n^2$.

31. From the sum of $a^4 + b^4$ and $4a^3b + 6a^2b^2 + 4ab^3$ take $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 - b^4$.

32. From the difference of $a^8 + b^8$ and $4a^2b^4 - 4a^4b^2$ take $a^8 - 4a^2b^4 - 4a^4b^2 - b^8$.

33. From $x^6 + 5x^5 + 4x^4 + 3x^3 + 2x^2 + x + 1$ take $x^6 + 5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 1$.

34. From $x^{4n} - 3x^{3n} + 2x^{2n} - x^n + 2x^2 - 3x^3 + 5$ take $x^{4n} + 3x^{3n} - 2x^{2n} - x^n + 2x^2 + 3x^3 + 8$.

35. From $8(a+b)^3 - 5(a+b)^2 + 12(a+b) + 10$ take $6(a+b)^3 - 5(a+b)^2 + 12(a+b) - 18$.

36. From the sum of $5\sqrt{a^2 - x^2} - 6\sqrt{b^2 - y^2}$ and $7\sqrt{a^2 - x^2} + 4\sqrt{b^2 - y^2}$ take $10\sqrt{a^2 - x^2} - 10\sqrt{b^2 - y^2}$.

37. From the sum of $2\sqrt[3]{x^2 - y^2} + 3\sqrt{x^2 - y^2}$ and $3\sqrt[3]{x^2 - y^2} - 2\sqrt{x^2 - y^2}$ take $6\sqrt[3]{x^2 - y^2} - 5\sqrt{x^2 - y^2}$.

38. From $mx + ny$ take $ax - by$.

Process.

$$\begin{array}{l} mx + ny \\ \underline{ax - by} \\ (m-a)x + (n+b)y, \text{ Ans.} \end{array}$$

a times x from m times x is evidently $(m - a)$ times x, or $(m - a)x$. The difference between n times y and $-b$ times y is evidently $(n + b)$ times y, or $(n + b)y$.

39. From $ax + by$ take $cx - dy$.

40. From $mx^2 - ny^2$ take $px^2 - qy^2$.

41. From $m(x - a) - n(y - b)$ take $(x - a) + (y - b)$.

42. From $a\sqrt{x + y} + b\sqrt{x - y}$ take $b\sqrt{x + y} - a\sqrt{x - y}$.

43. From $m(x^2 + 2x + 1)^2$ take $n(x^2 + 2x + 1)^2$.

44. From $a(x^2 + xy + y^2)^3$ take $-b(x^2 + xy + y^2)^3$.

45. From $(a^3 + b^3 + c^3)x$ take $(a^3 + b^3 - c^3 + 2)y$.

46. From $(x^2 + y^2)a - (x^2 - y^2)b$ take $(x^2 + y^2)b + (x^2 - y^2)a$.

47. From $3a\sqrt[n]{xy} + 4b\sqrt[m]{xy}$ take $4a\sqrt[n]{xy} - 3b\sqrt[m]{xy}$.

USE OF THE PARENTHESES.

64. The **Parentheses** are so frequently used in algebra that it is important that the principles which govern their use should be well understood.

PRINCIPLES.

1. *Parentheses preceded by a plus sign may be removed without changing the signs of the inclosed terms.*

The expression

$$4a - b + (mx + xy - y^2)$$

indicates that $mx + xy - y^2$ is to be added to $4a - b$. Adding these quantities, by ART. 56, Case 2d, we have

$$4a - b + mx + xy - y^2.$$

It is seen that the signs of the terms within the parentheses are *unchanged* when the parentheses are removed.

2. *A quantity may be inclosed in parentheses if it is preceded by the plus sign, without changing the signs of the inclosed terms.*

This is evident, being the converse of Prin. 1.

3. *Parentheses preceded by a minus sign may be removed if the sign of each inclosed term be changed.*

The expression

$$4a - b - (mx + xy - y^2)$$

indicates that $mx + xy - y^2$ is to be subtracted from $4a - b$. Subtracting these quantities, by ART. 63, we have

$$4a - b - mx - xy + y^2.$$

It is seen that the signs of the terms within the parentheses are all changed when the parentheses are removed.

4. *A quantity may be inclosed in parentheses when it is preceded by the minus sign, if the sign of each term inclosed be changed.*

This is evident, being the converse of Prin. 3.

65. Expressions sometimes occur containing two or more signs of aggregation; as, $a - \{b - (c - \overline{d+e})\}$.

Such signs may be removed in succession, by beginning with the *inside pair* or with *any pair*.

After the signs are removed, the terms should be collected.

EXAMPLES.

1. Remove the parentheses from

$$x - (a+b) + (c-d) - (m-n).$$

By Art. 64, Prin. 1 and 3, the expression becomes

$$x - a - b + c - d - m + n.$$

2. Simplify $a - \{b + (a - \overline{b-c})\}$.

We remove the vinculum first, and then the parentheses, and finally the braces.

$$\begin{aligned} &a - \{b + (a - \overline{b-c})\} \\ &= a - \{b + (a - b + c)\} \\ &= a - \{b + a - b + c\} \\ &= a - b - a + b - c. \end{aligned}$$

Collecting, equals $-c$, *Ans.*

Reduce the following expressions to their simplest forms:

3. $+(-a)$; $-(-a)$; $+(a-b)$; $-(a-b)$.

4. $a+(-b)$; $a-(-b)$; $x-(x-y)$.

5. $x-(-x-y)$; $x+y-(-x+y)$.

6. $a+b-(-a+2)$; $x-y-(-b-3)$.

7. $3x+2a-(2x-3a+5)$.

8. $7x-4y+z-(-6x-5y-z)$.

9. $5a+(-2b+a)-(6a-4b-2)$.

10. $1-(1-x)+(x-x^2)-(2x-3x^2)$.

11. $-(a-b\overline{+c})$; $+(a-\overline{b-c})$.

12. $a-(-b-\overline{c-b})$; $a-(b+\overline{c-b})$.

13. $-(-\overline{a-b})$; $-(+\overline{a-b})$; $-(-\overline{a+b})$.

14. $a-\{b-c+(a-b-\overline{b-e})\}$.

15. $x-[4a+b-\{6b-(-8d+2b-\overline{3b-a})\}]$.

16. $-[2a-\{b+(3c-2b+\overline{c-a-b})\}]$.

17. $x-[-4a-\{-6b-(-8d+\overline{6b-a+d})\}]$.

18. $-(4x-\overline{7y+4})-\{-2x-(3y-\overline{4y-x+4})\}$.

19. $8m-[a-b+\{m+a-(b-m)+b\}-m-b]$.

Express the following in binomials, and then in trinomials:

20. $a-2b+3c-5d+x-y$.

Thus, in binomials, $(a-2b)+(3c-5d)+(x-y)$.

And, in trinomials, $(a-2b+3c)-(5d-x+y)$.

21. $-m+n-p+7q-8d+4w$.

22. $4a-3b-2c-d-e-3f$.

23. $2x-3y+2z-v+5w-2$.

24. $a^2-b^2-c^2+d^3-e^3-f^2$.

25. $6a+4b-\overline{x-y}-e^2-\overline{m+n}+c^2$.

Express $a-b+c-d-e+f-g-h-i-j+h-l$ in parentheses, preserving the order of the letters—

26. Taking the terms *two* together.

27. Taking the terms *three* together.

28. Taking the terms *four* together.

29. Taking the terms *six* together.

TRANSPOSITION OF TERMS IN EQUATIONS.

66. The **First Member** of an equation is the quantity on the left of the sign of equality.

67. The **Second Member** of an equation is the quantity on the right of the sign of equality.

68. Transposition is the process of changing a quantity from one member of an equation to the other.

69. From the axioms of ART. 54 we derive the following GENERAL PRINCIPLE: *Both members of an equation may be increased, diminished, multiplied, or divided by the same quantity without destroying the equality.*

1. Solve the equation $8x - 5 = 6x + 7$.

PROCESS.

$$\begin{array}{l} 8x - 5 = 6x + 7 \\ \underline{\quad +5 = \quad\ +5} \\ 8x = 6x + 7 + 5 \\ \underline{6x = 6x \qquad\qquad} \\ 8x - 6x = 7 + 5 \\ \text{or } 2x = 12 \\ \qquad x = 6, \textit{ Ans.} \end{array}$$

Adding 5 to both members of the equation, we have $8x = 6x + 7 + 5$.

Subtracting $6x$ from both members of this equation, we have $8x - 6x = 7 + 5$.

Collecting the terms of this last equation, we have $2x = 12$, or $x = 6$.

By examining this process we see that in changing -5 and $+6x$ from one member of the equation to the other, their signs are changed. Hence we have the following

RULE.

A term may be transposed from one member of an equation to the other if at the same time the sign be changed.

EXAMPLES.

Transpose and find the value of x in the following equations:

2. $x + 8 = 10$.
3. $5x - 4 = 6$.
4. $6x - 12 = 24$.
5. $9x - 4 = 32$.
6. $10x + 14 = 44$.
7. $12x - 20 = 40$.
8. $8x - 5 = 35$.
9. $7x + 5 = 40$.
10. $9x - 8 = 64$.
11. $6x = 5x + 4$.
12. $7x = 18 - 2x$.
13. $5x = 12 + 4x$.
14. $12 = 4x - 8$.
15. $3x + 4 = 2x + 6$.
16. $5x - 6 = 3x + 4$.
17. $6x - 4 - 2x = 20$.

18. $8x = 5x + 40 - x$.

19. $6x + 5 = 4x + 7$.

20. $7x - 5 = 3x - 1$.

21. $10x - 8 = 9x + 7$.

22. $15x - 8 = 6x + 64$.

23. $20x - 120 = -20 + 8x$.

24. $140 - 9x = 12x - 7$.

25. $4x - 8 + 6x - 52 = 0$.

26. $12 - 2x - 6 = 22 - 4x$.

27. $9x + 18 = 27 - x + 1$.

PROBLEMS.

Solve the following problems:

28. Five times a number increased by 17 equals nine times the number diminished by 27; what is the number?

29. Three cities, A, B, and C, together have a population of 49,500; what is the population of each if B has 5,500 more inhabitants than C, and A has 5,500 more than B?

30. A man being asked his age replied that if his age were doubled, and then diminished by 46 years, it would be four-score and ten years; what was his age?

31. Divide $270 between A, B, and C so that A shall have three times as much as B, lacking $10; and C shall have four times as much as B, plus $40.

32. The sum of three numbers is 450; what are the numbers if the first is five times the second, and the third is 10 more than the first?

33. A retired farmer deposited his fortune of $62,000 in three different banks; how much is deposited in each bank if there are $5,000 more in the second than in the first, and $7,000 more in the third than in the second?

34. A horse, carriage, and harness together are worth $440; what is the cost of each if the horse cost fifteen times as much as the harness, and the carriage cost $180 less than the horse?

35. Divide the number 594 into three such parts that the second shall be 10 more than the first, and the third shall be 18 less than the sum of the other two parts.

MULTIPLICATION.

70. Multiplication is the process of taking one quantity as many times as there are units in another.

The **Multiplicand** is the quantity taken.

The **Multiplier** is the quantity which indicates how often the multiplicand is taken.

The **Product** is the result of the process.

The multiplicand and multiplier are *factors* of the *product.*

PRINCIPLES.

71.—1. *The product of two or more factors is the same in whatever order they may be taken.*

It is proved in arithmetic that 3×5 and 5×3 are each equal to 15. In like manner, in algebra, we have $a \times b$ and $b \times a$ each equal to ab. Hence

$$ab = ba.$$

2. *Multiplying any factor of a quantity multiplies the quantity.*

Thus, four times 6×3 equals 4×6 times 3, or six times 3×4, all of which equal 72. Also, five times $3a$ is $15a$.

3. *The exponent of a letter in the product is equal to the sum of its exponents in the two factors.*

Since the exponent of a quantity indicates the number of times it is to be used as a factor (ART. 6), then

$$a^4 = a \times a \times a \times a, \text{ and } a^2 = a \times a. \quad \text{Hence}$$

$$a^4 \times a^2 = aaaa \times aa = a^6.$$

And in general, $\quad a^n \times a^m = a^{n+m}.$

4. *The product of two factors having like signs is positive, and having unlike signs is negative.*

Four possible cases may arise.

1st. Multiply $+a$ by $+b$.

This means that $+a$ is to be taken b times. Since multiplication is an abbreviated form of addition, and the sum of any number of positive quantities is positive, then $a \times b$ is $+ab$.

2d. Multiply $-a$ by b.

This means that $-a$ is to be taken b times. Since the sum of any number of negative quantities is negative, then $-a \times b$ is $-ab$.

3d. Multiply a by $-b$.

This means that a is to be taken b times, and the product is to be taken negatively. $a \times b$ is ab (Case 1), and taken negatively is $-ab$.

4th. Multiply $-a$ by $-b$.

This means that $-a$ is to be taken b times, and the product is to be taken negatively. $-a \times b$ is $-ab$ (Case 2), and taken negatively is $-(-ab) = +ab$.

Hence the product of

Like signs gives $+$, and of
Unlike signs gives $-$.

5. *Multiplying each term of a quantity multiplies the entire quantity.*

Thus, $(a+b)2 = (a+b) + (a+b)$; or, removing the parentheses, we have $a + a + b + b$, which equals $2a + 2b$.

And in general $(a+b)n = an + bn$.

72. To multiply when the multiplier is a monomial.

1. Multiply $4a^2x$ by $2a^3$.

Process.	To multiply $4a^2x$ by $2a^3$ we must multiply by 2 and a^3.
$4a^2x$	Two times $4a^2x$ is $8a^2x$ (Prin. 2 and 4).
$\underline{2a^3}$	a^3 times $8a^2x$ is $8a^5x$ (Prin. 2 and 3).
$8a^5x$, *Ans.*	Therefore, $4a^2x$ multiplied by $2a^3$ is $8a^5x$.

2. Multiply $3a^3 - 2a^2b^3$ by $3a^2b$.

Process.

$$\begin{array}{l} 3a^3 - 2a^2b^3 \\ \underline{3a^2b \qquad\qquad} \\ 9a^5b - 6a^4b^4, \textit{ Ans.} \end{array}$$

Since multiplying each term of a quantity multiplies the entire quantity (Prin. 5), we multiply each term of $3a^3 - 2a^2b^3$ by $3a^2b$, and obtain $9a^5b - 6a^4b^4$ (Ex. 1).

From these examples we derive the following

RULE.

Multiply each term of the multiplicand by the multiplier. To the product of the coefficients annex the literal factors, giving each letter an exponent equal to the sum of its exponents in both factors. Make the product positive when the factors have like signs, and negative when they have unlike signs.

EXAMPLES.

Multiply—

3.	4.	5.	6.	7.
4	-4	4	-4	x
2	2	-2	-2	-2
8.	9.	10.	11.	12.
x^2	x^6	$-a^{10}$	$-y^5$	m^4
x^4	$-x^9$	a^4	$-y^8$	m^{11}
13.	14.	15.	16.	17.
x^n	$-a^{2n}$	$-a^{3n}$	$-x^m$	y^{n+3}
$-x^n$	a^n	$-a^{4n}$	x^n	$-y^n$
18.	19.	20.	21.	22.
$-a^{m+4}$	$-x^{m-6}$	y^{n-3}	$-a^{n^2}$	x^{m^2}
a^{m-3}	$-x^{m+8}$	y^{m-2}	$-a^{n^2}$	$-x^{m^3}$

Multiply—

23. $6x^2y^3$ by $-3x^3y^4$.
24. $-5a^3b^5$ by $-4ab^4$.
25. $3x^n$ by $-ax^n$.
26. $-4am^n$ by $-3a^mm^2$.
27. $6xy^n$ by $-5x^ny$.
28. $-8a^{m-2}$ by $-5a^{m+3}$.
29. $2(x+y)^2$ by $-4(x+y)^3$.
30. $(a+b)^n$ by $(a+b)^2$.
31. $-(x-y)^4$ by $(x-y)^6$.
32. $(m+n)^5$ by $-(m+n)^5$.
33. $(a-x)^n$ by $-(a-x)^m$.
34. $6x^{2n-2}$ by $-2x^{2-n}$.
35. $7a^{2-n}$ by $5a^{3+n}$.
36. $-\frac{1}{2}(x+y)^4$ by $\frac{1}{2}(x+y)^8$.

Find the value of—

37. $-a^2 \times -a^3 \times -a^4$.

38. $-a \times -a^2 \times -a^3 \times -a^4$.

39. $-x^3 \times -x^3 \times -x^3 \times -x^3 \times -x^3$.

40. $-y^n \times -y^n \times -y^n \times -y^n \times -y^n \times -y^n$.

41. $-a^n \times -a^{2n} \times -a^{3n} \times -a^{4n} \times -a^{5n}$.

42. $a(m+n)^x \times a^2(m+n)^{2x} \times a^3(m+n)^{3x}$.

43. $a^{n+1} \times -a^{n-2} \times a^{n-3} \times -a^{n+4} \times a^{3n}$.

44. $(a+x)^{m+2} \times (a+x)^{m-4} \times (a+x)^{m+2}$.

45. $3^x \times 3^{x+1} \times 3^{x-2} \times 3^{x-4} \times 3^{4-2x}$.

46. $4^n \times -4^{2n} \times 4^{3n} \times -4^{4n} \times 4^{5n} \times -4^{6n}$.

EXAMPLES.

Multiply—

47.	48.	49.	50.
$3x^2 - 2y$	$a + b - c$	$2m^2 - n^3$	$x^2 - y^2$
$-2x$	$-3abc$	$4mn$	$3x^n$
$-6x^3 + 4xy$, *Ans.*			

51. $a^2 + b^3 - c^n$ by $4ac^2$.

52. $x^2 + y^2$ by $5xy$.

53. $a^n - b^n$ by $4a^nb^n$.

54. $3x^2 - 2y^2$ by $6x^3y^3$.

55. $x^n + xy + y^n$ by $2xy$.

56. $4ax^2 - 3y^3$ by $-2x^3y^4$.

57. $a^2 + 2ab + b^2$ by a^2b^2.

58. $x^{3n} + y^{2n}$ by $-5x^ny^n$.

59. $m^4 - m^2n^2 + n^2$ by mn.

60. $x^2 - y^2 + z^2$ by x^2yz^3.

61. $x^4 - y^4$ by $4x^6y^8$.

62. $a^n - b^n$ by $3a^2b^2$.

63. $x^2 - 2xy + y^2$ by $3xy$.

64. $x^2 + x + 1$ by $-4x^3$.

65. $a^3 - 3 + a^2$ by na^n.

66. $a^n - ab + b^n$ by $a^{2n}b^{3n}$.

67. $x^8 - x^4y^4 + b^8$ by $-x^2y^2$.

68. $y^3 - 3y^2 - y$ by $-4y^5$.

69. $x^3 + x^2 - x$ by ax^2.

70. $a^4 - a^2b^2 + b^4$ by $-ab$.

71. $3x^n - x^2y^2 - y^n$ by $2x^ny^n$.

72. $\frac{1}{2}x^2 - \frac{1}{3}xy + \frac{1}{2}y^2$ by $6xy$.

73. $4x^2 - 3ax - 2a^2$ by $-3a^2x^2$.

74. $5a^n - 4ab + 5b^n$ by $-3ab$.

73. To multiply when the multiplier is a polynomial.

1. Multiply $2a - 3b$ by $a + 2b$.

PROCESS.

$$\begin{array}{rrl} 2a & -\ 3b & \\ a & +\ 2b & \\ \hline 2a^2 & -\ 3ab & \\ & +\ 4ab & -\ 6b^2 \\ \hline 2a^2 & +\ ab & -\ 6b^2, \textit{ Ans.} \end{array}$$

$a + 2b$ times $2a - 3b$ equals a times $2a - 3b$ plus $2b$ times $2a - 3b$. a times $2a - 3b$ is $2a^2 - 3ab$ (Art. 72), and $2b$ times $2a - 3b$ is $4ab - 6b^2$. Adding the partial products, we have $2a^2 + ab - 6b^2$.

From this example we derive the following

RULE.

Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.

EXAMPLES.

2.

$$\begin{array}{lll} m & +\ n & \\ m & +\ n & \\ \hline m^2 & +\ mn & \\ & \ mn & +\ n^2 \\ \hline m^2 & +\ 2mn & +\ n^2, \textit{ Ans.} \end{array}$$

3.

$$\begin{array}{lll} m & -\ n & \\ m & -\ n & \\ \hline m^2 & -\ mn & \\ & -\ mn & +\ n^2 \\ \hline m^2 & -\ 2mn & +\ n^2 \end{array}$$

4.

$$\begin{array}{lll} m & +\ n & \\ m & -\ n & \\ \hline m^2 & +\ mn & \\ & -\ mn & -\ n^2 \\ \hline m^2 & & -\ n^2 \end{array}$$

5.

$$\begin{array}{lllll} a^2 & +\ ab & +\ b^2 & & \\ a^2 & -\ ab & +\ b^2 & & \\ \hline a^4 & +\ a^3b & +\ a^2b^2 & & \\ & -\ a^3b & -\ a^2b^2 & -\ ab^3 & \\ & & \ a^2b^2 & +\ ab^3 & +\ b^4 \\ \hline a^4 & & +\ a^2b^2 & & +\ b^4 \end{array}$$

6.

$$\begin{array}{lllll} x^4 & -\ x^2 & +\ 1 & & \\ x^4 & +\ x^2 & +\ 1 & & \\ \hline x^8 & -\ x^6 & +\ x^4 & & \\ & +\ x^6 & -\ x^4 & +\ x^2 & \\ & & \ x^4 & -\ x^2 & +\ 1 \\ \hline x^8 & & +\ x^4 & & +\ 1 \end{array}$$

Multiply—

7. $x + y$ by $x - y$.
8. $x^2 - y^2$ by $x^2 + y^2$.
9. $x + y$ by $x + y$.
10. $x - y$ by $x - y$.
11. $3x - 2y$ by $2x + 3y$.
12. $4x + 3y$ by $4x - 3y$.
13. $5x + 2y$ by $5x + 2y$.
14. $2x - 3a$ by $2x + 3a$.

15. $5x + 2a$ by $5x + 2a$.
16. $a^4 - b^4$ by $a^3 - b^3$.
17. $ax + ay$ by $ax - ay$.
18. $3x + 2y$ by $4x - 3y$.
19. $a^2 + x^2$ by $a^3 - x^4$.
20. $a^6 + y^6$ by $a^6 + y^6$.
21. $a^2 + a + 1$ by $a - 1$.
22. $a^2 - a + 1$ by $a + 1$.
23. $x^4 + x^2y^2 + y^4$ by $x^2 - y^2$.
24. $a^5 + a^4 + a^3$ by $a - 1$.
25. $a^6 - a^5 + a^4$ by $a + 1$.
26. $a^2 + ab + b^2$ by $a - b$.
27. $a^6 + a^4 + a^2$ by $a^2 - 1$.
28. $a^4 - a^2b^2 + b^4$ by $a^2 + b^2$.
29. $a^2 + a + 1$ by $a^2 - a + 1$.
30. $x^2 - xy + y^2$ by $x^2 + xy + y^2$.
31. $1 - 2a + 4a^2$ by $1 + 2a + 4a^2$.
32. $x^3 + x^2 + x + 1$ by $x - 1$.
33. $x^4 - x^3 + x^2 - x + 1$ by $x + 1$.
34. $x^4 - x^2y^3 + y^6$ by $x^2 + y^3$.
35. $a^2 + ax + x^2$ by $a^2 - ax + x^2$.
36. $x^5 + x^4 + x^3 + x^2 + x + 1$ by $x - 1$.
37. $a^n - b^n$ by $a^n + b^n$.
38. $x^n + y^n$ by $x^n - y^n$.
39. $x^n - y^n$ by $x^n - y^n$.
40. $a^n + b^n$ by $a^n + b^n$.
41. $x^n - y^m$ by $x^m + y^n$.
42. $a^x - b^y$ by $a^x + b^y$.
43. $m^m - n^n$ by $m^m + n^n$.
44. $a^{n+1} + b^{n-1}$ by $a^{n+1} - b^{n-1}$
45. $x^{2n} - y^{2n}$ by $x^{2n} - y^{2n}$.
46. $y^{3n} + x^{4n}$ by $y^{3n} + x^{4n}$.
47. $x^4 - x^2y^2 + y^4$ by $x^2 + y^2$.
48. $x^2 + xy + y^2$ by $x^2 + xy - y^2$.
49. $x^{2n} - x^ny^n + y^{2n}$ by $x^{2n} + x^ny^n + y^{2n}$.
50. $a^{m+n} - b^{m-n}$ by $a^{m+n} + b^{m-n}$.
51. $a^{n+2} + b^{n+2}$ by $a^{n+2} - b^{n+2}$.
52. $x^{m-n} - y^{m+n}$ by $x^{m-n} + y^{m+n}$.
53. $a^{m+n+1} + b^{m+n-1}$ by $a^{m+n+1} - b^{m+n-1}$.

Expand the following:

54. $(m-a)(m+a)$.

55. $(m-a)(m+a)(m^2+a^2)$.

56. $(m+a)(m^2+a^2)(m-a)(m^4+a^4)$.

57. $(n^2-np+p^2)(n^2+np+p^2)$.

58. $(x+y)(x-y)(x+y)(x-y)$.

59. $(a^2+a+1)(a^2-a+1)(a+1)(a-1)$.

60. $(a^4-a^2y^2+y^4)(a^2+y^2)$.

61. $(x^2-x+1)(x^2+x+1)(x^4-x^2+1)$.

EQUATIONS AND PROBLEMS.

74. A **Numerical Equation** is one in which the known quantities are expressed by numbers; as,

$$3x+4=2x+12.$$

75. A **Literal Equation** is one in which some or all of the known quantities are expressed by letters; as,

$$4x-2x=a+4.$$

$$\text{Or,}\quad 4x+2x=a+b.$$

76. An **Identical Equation** is one in which the two members are the same, or will become the same when the operations indicated are performed; as,

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$$\text{Or,}\quad 3(x+5)=3x+15.$$

77. The **Root** of an equation is the value of the unknown quantity.

The root is said to be *verified*, and the equation to be *satisfied*, when the value of the root is substituted for the unknown quantity and the equation is reduced to identity.

Thus, in the equation $2x+6=16$, 5 is the root. If 5 be substituted for x, the equation will become $10+6=16$, an identical equation. Here the root is verified, and the equation is satisfied.

78. Solve the following equations, and verify the result:

1. $5(x+b)=3(x+a)$.

SOLUTION.

$5(x+b)=3(x+a).$

Expanding, $5x+5b=3x+3a.$

Transposing, $2x=3a-5b.$

Dividing by 2, $x=\frac{3a-5b}{2}$, *Ans.*

2. $5(x-2)=30$.
3. $4(3x+5)=80$.
4. $3(2x-4+x)=69$.
5. $4x=48-3(x+2)$.
6. $5x-2(2x-8)=12$.
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Solve the following problems, and verify the result:

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22. A, B, and C own 350 acres of land: if B had 10 acres more, A would have two times, and C three times, as much as B would then have; how many acres has each?

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Let $x =$ what each paid by the first condition.
Then $8x =$ the cost of the coach.
Let $x + \$2 =$ what each one paid by the second condition.
Then $6(x + \$2) =$ the cost of the coach.
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DIVISION.

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The **Dividend** is the quantity to be divided.

The **Divisor** is the quantity by which we divide.

The **Quotient** is the result of the process.

PRINCIPLE I.

80. *Taking a factor out of a quantity divides the quantity by that factor.*

If x is taken out of $3xy$ we have $3y$, which is $3xy$ divided by x, since $3y$ multiplied by x equals $3xy$.

PRINCIPLE II.

The exponent of a factor in the quotient equals its exponent in the dividend minus its exponent in the divisor.

Thus, $\frac{a^5}{a^3} = \frac{aaaaa}{aaa} = aa = a^2$ (Prin. 1).

Hence $a^5 \div a^3 = a^{5-3} = a^2$.

Also, $a^{10} \div a^4 = a^{10-4} = a^6$.

And, in general, $a^m \div a^n = a^{m-n}$.

PRINCIPLE III.

The quotient is positive when the dividend and divisor have like signs, and negative when they have unlike signs; or, in general, *the quotient of* **like** *signs gives* **plus,** *and of* **unlike** *signs,* **minus.**

Thus, $+ab \div b = +a$, since $+a \times +b = +ab$.

$-ab \div b = -a$, since $-a \times +b = -ab$.

$+ab \div -b = -a$, since $-a \times -b = +ab$.

$-ab \div -b = +a$, since $+a \times -b = -ab$.

PRINCIPLE IV.

Any quantity whose exponent is zero is equal to 1.

Thus, $\frac{a^4}{a^4} = a^{4-4} = a^0$ (Prin. 2).

$\frac{a^4}{a^4} = 1$. Hence $a^0 = 1$ (Art. 54, Ax. 9).

From these examples we derive the following

RULE.

Multiply each term of the multiplicand by the multiplier. To the product of the coefficients annex the literal factors, giving each letter an exponent equal to the sum of its exponents in both factors. Make the product positive when the factors have like signs, and negative when they have unlike signs.

EXAMPLES.

Multiply—

3.	4.	5.	6.	7.
4	-4	4	-4	x
2	2	-2	-2	-2

8.	9.	10.	11.	12.
x^2	x^6	$-a^{10}$	$-y^5$	m^4
x^4	$-x^9$	a^4	$-y^8$	m^{11}

13.	14.	15.	16.	17.
x^n	$-a^{2n}$	$-a^{3n}$	$-x^m$	y^{n+3}
$-x^n$	a^n	$-a^{4n}$	x^n	$-y^n$

18.	19.	20.	21.	22.
$-a^{m+4}$	$-x^{m-6}$	y^{n-3}	$-a^{n^2}$	x^{m^2}
a^{m-3}	$-x^{m+8}$	y^{m-2}	$-a^{n^2}$	$-x^{m^3}$

Multiply—

23. $6x^2y^3$ by $-3x^3y^4$.
24. $-5a^3b^5$ by $-4ab^4$.
25. $3x^n$ by $-ax^n$.
26. $-4am^n$ by $-3a^mm^2$.
27. $6xy^n$ by $-5x^ny$.
28. $-8a^{m-2}$ by $-5a^{m+3}$.
29. $2(x+y)^2$ by $-4(x+y)^3$.
30. $(a+b)^n$ by $(a+b)^2$.
31. $-(x-y)^4$ by $(x-y)^6$.
32. $(m+n)^5$ by $-(m+n)^5$.
33. $(a-x)^n$ by $-(a-x)^m$.
34. $6x^{2n-2}$ by $-2x^{2-n}$.
35. $7a^{2-n}$ by $5a^{3+n}$.
36. $-\frac{1}{2}(x+y)^4$ by $\frac{1}{2}(x+y)^8$.

Find the value of—

37. $-a^2 \times -a^3 \times -a^4$.

38. $-a \times -a^2 \times -a^3 \times -a^4$.

39. $-x^3 \times -x^3 \times -x^3 \times -x^3 \times -x^3$.

40. $-y^n \times -y^n \times -y^n \times -y^n \times -y^n \times -y^n$.

41. $-a^n \times -a^{2n} \times -a^{3n} \times -a^{4n} \times -a^{5n}$.

42. $a(m+n)^x \times a^2(m+n)^{2x} \times a^3(m+n)^{3x}$.

43. $a^{n+1} \times -a^{n-2} \times a^{n-3} \times -a^{n+4} \times a^{3n}$.

44. $(a+x)^{m+2} \times (a+x)^{m-4} \times (a+x)^{m+2}$.

45. $3^x \times 3^{x+1} \times 3^{x-2} \times 3^{x-4} \times 3^{4-2x}$.

46. $4^n \times -4^{2n} \times 4^{3n} \times -4^{4n} \times 4^{5n} \times -4^{6n}$.

EXAMPLES.

Multiply—

47.	48.	49.	50.
$3x^2 - 2y$	$a + b - c$	$2m^2 - n^3$	$x^2 - y^2$
$-2x$	$-3abc$	$4mn$	$3x^n$
$-6x^3 + 4xy$, *Ans.*			

51. $a^2 + b^3 - c^n$ by $4ac^2$.

52. $x^2 + y^2$ by $5xy$.

53. $a^n - b^n$ by $4a^nb^n$.

54. $3x^2 - 2y^2$ by $6x^3y^3$.

55. $x^n + xy + y^n$ by $2xy$.

56. $4ax^2 - 3y^3$ by $-2x^3y^4$.

57. $a^2 + 2ab + b^2$ by a^2b^2.

58. $x^{3n} + y^{2n}$ by $-5x^ny^n$.

59. $m^4 - m^2n^2 + n^2$ by mn.

60. $x^2 - y^2 + z^2$ by x^2yz^3.

61. $x^4 - y^4$ by $4x^6y^8$.

62. $a^n - b^n$ by $3a^2b^2$.

63. $x^2 - 2xy + y^2$ by $3xy$.

64. $x^2 + x + 1$ by $-4x^3$.

65. $a^3 - 3 + a^2$ by na^n.

66. $a^n - ab + b^n$ by $a^{2n}b^{3n}$.

67. $x^8 - x^4y^4 + b^8$ by $-x^2y^2$.

68. $y^3 - 3y^2 - y$ by $-4y^5$.

69. $x^3 + x^2 - x$ by ax^2.

70. $a^4 - a^2b^2 + b^4$ by $-ab$.

71. $3x^n - x^2y^2 - y^n$ by $2x^ny^n$.

72. $\frac{1}{2}x^2 - \frac{1}{3}xy + \frac{1}{2}y^2$ by $6xy$.

73. $4x^2 - 3ax - 2a^2$ by $-3a^2x^2$.

74. $5a^n - 4ab + 5b^n$ by $-3ab$.

73. To multiply when the multiplier is a polynomial.

1. Multiply $2a - 3b$ by $a + 2b$.

PROCESS.

$$\begin{array}{rrr} 2a & -3b & \\ a & +2b & \\ \hline 2a^2 & -3ab & \\ & +4ab & -6b^2 \\ \hline 2a^2 & +ab & -6b^2, \textit{ Ans.} \end{array}$$

$a + 2b$ times $2a - 3b$ equals a times $2a - 3b$ plus $2b$ times $2a - 3b$. a times $2a - 3b$ is $2a^2 - 3ab$ (Art. 72), and $2b$ times $2a - 3b$ is $4ab - 6b^2$. Adding the partial products, we have $2a^2 + ab - 6b^2$.

From this example we derive the following

RULE.

Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.

EXAMPLES.

2.

$$\begin{array}{rrr} m & +n & \\ m & +n & \\ \hline m^2 & +mn & \\ & mn & +n^2 \\ \hline m^2 & +2mn & +n^2, \textit{ Ans.} \end{array}$$

3.

$$\begin{array}{rrr} m & -n & \\ m & -n & \\ \hline m^2 & -mn & \\ & -mn & +n^2 \\ \hline m^2 & -2mn & +n^2 \end{array}$$

4.

$$\begin{array}{rrr} m & +n & \\ m & -n & \\ \hline m^2 & +mn & \\ & -mn & -n^2 \\ \hline m^2 & & -n^2 \end{array}$$

5.

$$\begin{array}{rrrrr} a^2 & +ab & +b^2 & & \\ a^2 & -ab & +b^2 & & \\ \hline a^4 & +a^3b & +a^2b^2 & & \\ & -a^3b & -a^2b^2 & -ab^3 & \\ & & a^2b^2 & +ab^3 & +b^4 \\ \hline a^4 & & +a^2b^2 & & +b^4 \end{array}$$

6.

$$\begin{array}{rrrrr} x^4 & -x^2 & +1 & & \\ x^4 & +x^2 & +1 & & \\ \hline x^8 & -x^6 & +x^4 & & \\ & +x^6 & -x^4 & +x^2 & \\ & & x^4 & -x^2 & +1 \\ \hline x^8 & & +x^4 & & +1 \end{array}$$

Multiply—

7. $x + y$ by $x - y$.

8. $x^2 - y^2$ by $x^2 + y^2$.

9. $x + y$ by $x + y$.

10. $x - y$ by $x - y$.

11. $3x - 2y$ by $2x + 3y$.

12. $4x + 3y$ by $4x - 3y$.

13. $5x + 2y$ by $5x + 2y$.

14. $2x - 3a$ by $2x + 3a$.

15. $5x + 2a$ by $5x + 2a$.
16. $a^4 - b^4$ by $a^3 - b^3$.
17. $ax + ay$ by $ax - ay$.
18. $3x + 2y$ by $4x - 3y$.
19. $a^2 + x^2$ by $a^3 - x^4$.
20. $a^6 + y^6$ by $a^6 + y^6$.
21. $a^2 + a + 1$ by $a - 1$.
22. $a^2 - a + 1$ by $a + 1$.
23. $x^4 + x^2y^2 + y^4$ by $x^2 - y^2$.
24. $a^5 + a^4 + a^3$ by $a - 1$.
25. $a^6 - a^5 + a^4$ by $a + 1$.
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Expand the following:

54. $(m-a)(m+a)$.

55. $(m-a)(m+a)(m^2+a^2)$.

56. $(m+a)(m^2+a^2)(m-a)(m^4+a^4)$.

57. $(n^2-np+p^2)(n^2+np+p^2)$.

58. $(x+y)(x-y)(x+y)(x-y)$.

59. $(a^2+a+1)(a^2-a+1)(a+1)(a-1)$.

60. $(a^4-a^2y^2+y^4)(a^2+y^2)$.

61. $(x^2-x+1)(x^2+x+1)(x^4-x^2+1)$.

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$-ab \div -b = +a$, since $+a \times -b = -ab$.

PRINCIPLE IV.

Any quantity whose exponent is zero is equal to 1.

Thus, $\frac{a^4}{a^4} = a^{4-4} = a^0$ (Prin. 2).

$\frac{a^4}{a^4} = 1$. Hence $a^0 = 1$ (Art. 54, Ax. 9).

PRINCIPLE V.

Any quantity, with either a negative or a positive exponent, is equal to the reciprocal of itself with the sign of its exponent changed.

Thus, $\frac{a^4}{a^6} = a^{4-6} = a^{-2}$ (Prin. 2); and $\frac{a^4}{a^6} = \frac{1}{a^2}$, dividing by a^4.

Hence $a^{-2} = \frac{1}{a^2}$ (ART. 54, Ax. 9).

Also, $\frac{a^6}{a^4} = a^{6-4} = a^2$ (Prin. 2); and $\frac{a^6}{a^4} = \frac{1}{a^{-2}}$, dividing by a^6.

Hence $a^2 = \frac{1}{a^{-2}}$ (ART. 54, Ax. 9).

PRINCIPLE VI.

Dividing each term of a quantity divides the quantity.

Since $(a + b)c = ac + bc$, it follows that $(ac + bc) \div c = a + b$.

81. To divide when the divisor is a monomial.

1. Divide $6a^5$ by $2a^3$.

PROCESS.

$$2a^3\overline{)6a^5}$$
$$3a^2, \textit{ Ans.}$$

To divide $6a^5$ by $2a^3$, we must take out the factors 2 and a^3 (Prin. 1).

6 divided by 2 equals 3, and a^5 divided by a^3 equals a^2 (Prin. 2).

Hence $6a^5$ divided by $2a^3$ equals $3a^2$.

2. Divide $9a^5b^2 - 6a^4b^5$ by $3a^2b^2$.

PROCESS.

$$3a^2b^2\overline{)9a^5b^2 - 6a^4b^5}$$
$$3a^3 \quad - 2a^2b^3, \textit{ Ans.}$$

Since dividing each term of a quantity divides the quantity (Prin. 6), we divide each term of $9a^5b^2 - 6a^4b^5$ by $3a^2b^2$, and obtain $3a^3 - 2a^2b^3$.

From these examples we derive the following

RULE.

Divide each term of the dividend by the divisor. To the quotient of the coefficients annex the literal quantities, giving each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor. Make the quotient + when the dividend and divisor have like signs, and — when they have unlike signs.

EXAMPLES.

3. $2\underline{)4}$

4. $2\underline{)-4}$

5. $-2\underline{)4}$

6. $-2\underline{)-4}$

7. $a\underline{)a^2}$

8. $x^2\underline{)x^4}$

9. $-x^6\underline{)x^9}$

10. $x^4\underline{)-x^7}$

11. $-a^8\underline{)a^{14}}$

12. $-n^4\underline{)-n^{10}}$

13. $a^n\underline{)-a^{2n}}$

14. $-x^{2n}\underline{)-x^{6n}}$

15. $-a^{5n}\underline{)a^{8n}}$

16. $x^2\underline{)-x^{5n}}$

17. $-x^n\underline{)x^8}$

18. $-m^2\underline{)m^n}$

19. $-a^{n-2}\underline{)a^{n+3}}$

20. $a^{m+2}\underline{)a^{m-3}}$

21. $-x^4\underline{)-x^n}$

22. $y^{n+1}\underline{)y^{n-1}}$

Divide—

23. $8a^2x^4$ by $4ax^2$.

24. $12n^2x^3$ by $-6n^2x$.

25. $-9x^4y^4$ by $-3x^2y^3$.

26. $-10y^4a^8$ by $5y^4a^6$.

27. $15a^2b^3c^4$ by $3ab^2c$.

28. $-8a^8b^{12}$ by $-4a^4b^8$.

29. $25m^2n^{10}$ by $5mn^5$.

30. $18x^2y^3b$ by $-6xy^3$.

31. $-15n^4x^3y^2$ by $5n^4x$.

32. $12x^7y^3z^2$ by $-6xz^2$.

33. $-a^n$ by a^m.

34. a^{m+n} by a^{2n}.

35. x^{m-n} by x^{n-m}.

36. x^{5m-4} by $-x^{6-m}$.

37. $12x^{m+n}$ by $-2x^{m-n}$.

38. $a^{3m}b^{2n}$ by $-a^4b^{3m}$.

39. $(a+b)^5$ by $(a+b)^3$.

40. $(a+b)^n$ by $(a+b)^m$.

41. $(x+y)^{m+n}$ by $(x+y)^{m-n}$.

42. $(a-b)^{x-y}$ by $(a-b)^{x+y}$.

43. $8a^2b^3-12ab^2$ by $4ab$.

44. $15a^3n^2-20a^2m^3$ by $-5a^2$.

45. $42x^3y^2-18x^2y^3$ by $6x^2y^2$.

46. $8m^4n-6mn^4$ by $-2mn$.

47. $7x^5-5x^4-4x^3$ by $-x^2$.

48. $34x^6y^4-51x^4y^6$ by $17x^4y^4$.

49. $100x^2-50x$ by $25x$.

50. $x^2+2xy+yx^2$ by $-x$.

51. $m^2n+mn^2+n^3$ by $-n$.

52. $12x^2n^4-6x^4n$ by $3x^2n^2$.

53. $16a^4m^8+10a^5m^9$ by $\frac{1}{2}a^4m^9$.

54. $\frac{1}{2}a^4y^6-6a^6by^4$ by $-\frac{1}{3}a^3y^3$.

55. $6m^{10}n^8-8m^{11}n^{12}$ by $\frac{1}{2}m^4n^6$.

56. $50m^4x^9-9m^4x^5$ by $-\frac{1}{3}m^4x^3$.

57. $18a^2b^4c^3 - 24a^4b^3c - 30a^2b^3c^4$ by $6a^2b^2c$.

58. $x^2y^2 + 2xy^3 + 3xy^2 - 4x^3y^2$ by $-xy$.

59. $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ by x^3.

60. $6a^{4n} -- 12a^{3n} + 18a^{2n} - 24a^n + 30$ by $6a^{2n}$.

61. $y^{n+4} - y^{n+3} + y^{n+2} + y^{n+1} + y$ by y.

62. $x^{n+1} + x^{n+2} + x^{n+3} + x^{n+4}$ by x^3.

63. $6(a + b) - 18(a + b)^2 + 12(a + b)^4$ by $-3(a + b)^2$.

64. $3a(x + y)^n + 6a(x + y)^{2n} - a(x + y)^{3n}$ by $a(x + y)^n$.

82. To divide when the divisor is a polynomial.

1. Divide $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.

PROCESS.

$$
\begin{array}{r|l}
 & \underline{a + b}\text{, Divisor.} \\
a^3 + 3a^2b + 3ab^2 + b^3 & \underline{a^2 + 2ab + b^2}\text{, Quotient.} \\
\underline{a^3 + a^2b \phantom{{}+ 3ab^2 + b^3}} & \\
2a^2b + 3ab^2 \phantom{{}+ b^3} & \\
\underline{2a^2b + 2ab^2 \phantom{{}+ b^3}} & \\
ab^2 + b^3 & \\
ab^2 + b^3 &
\end{array}
$$

For convenience of multiplying, write the divisor at the right of the dividend, and arrange both dividend and divisor according to the descending (or ascending) powers of a.

Since the first term of the dividend, a^3, is equal to the product of the first term of the divisor, a, by the first term of the quotient, we can find the first term of the quotient by dividing a^3 by a. a is contained in a^3, a^2 times; a^2 times $a + b$ equals $a^3 + a^2b$. Subtracting, and bringing down the next term, we have $2a^2b + 3ab^2$.

Since the first term of this new dividend, $2a^2b$, is equal to the product of the first term of the divisor, a, by the second term of the quotient, we can find the second term of the quotient by dividing $2a^2b$ by a. a is contained in $2a^2b$, $2ab$ times; $2ab$ times $a + b$ equals $2a^2b + 2ab^2$. Subtracting, and bringing down the next term, we have $ab^2 + b^3$.

Since the first term of this new dividend, ab^2, is equal to the product of the first term of the divisor, a, by the third term of the quotient, we can find the third term of the quotient by dividing ab^2 by a. a is contained in ab^2, b^2 times; b^2 times $a + b$ equals $ab^2 + b^3$. Subtracting, there is nc remainder. Hence the quotient is $a^2 + 2ab + b^2$.

From this example we derive the following

RULE.

Arrange both dividend and divisor according to the ascending or descending powers of a common letter.

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Multiply the divisor by the first term of the quotient, and subtract the product from the dividend.

Treat the remainder as a new dividend, and proceed as before.

Continue the process until the remainder does not contain the leading letter.

If there is a remainder after the last division, write it over the denominator and annex it with its proper sign to the quotient.

EXAMPLES.

2.
$$\begin{array}{l|l} a^3 - b^3 & a - b \\ \cline{2-2} \underline{a^3 - a^2b} & \underline{a^2 + ab + b^2} \\ \quad a^2b - b^3 & \\ \quad \underline{a^2b - ab^2} & \\ \qquad ab^2 - b^3 & \\ \qquad \underline{ab^2 - b^3} & \end{array}$$

3.
$$\begin{array}{l|l} x^4 - a^4 & x - a \\ \cline{2-2} \underline{x^4 - x^3a} & \underline{x^3 + x^2a + xa^2 + a^3} \\ \quad x^3a - a^4 & \\ \quad \underline{x^3a - x^2a^2} & \\ \qquad x^2a^2 - a^4 & \\ \qquad \underline{x^2a^2 - xa^3} & \\ \qquad\quad xa^3 - a^4 & \\ \qquad\quad \underline{xa^3 - a^4} & \end{array}$$

4.
$$\begin{array}{l|l} x^4 + x^2y^2 + y^4 & x^2 + xy + y^2 \\ \cline{2-2} \underline{x^4 + x^3y + x^2y^2} & \underline{x^2 - xy + y^2} \\ \quad - x^3y + y^4 & \\ \quad \underline{- x^3y - x^2y^2 - xy^3} & \\ \qquad x^2y^2 + xy^3 + y^4 & \\ \qquad \underline{x^2y^2 + xy^3 + y^4} & \end{array}$$

5.
$$\begin{array}{l|l}
a^n - b^n & a - b \\ \cline{2-2}
a^n - a^{n-1}b & a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \text{etc.} \\ \cline{1-1}
\quad a^{n-1}b - b^n & \\
\quad \underline{a^{n-1}b - a^{n-2}b^2} & \\
\qquad a^{n-2}b^2 - b^n & \\
\qquad \underline{a^{n-2}b^2 - a^{n-3}b^3} & \\
\qquad\quad a^{n-3}b^3 - b^n, \text{ etc.} &
\end{array}$$

Divide—

6. $x^2 + 2xy + y^2$ by $x + y$.

7. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.

8. $a^2 - 6ax + 9x^2$ by $a - 3x$.

9. $16x^4 - 8x^2a^2 + a^4$ by $4x^2 - a^2$.

10. $a^3 + a^2b + ab^2 + b^3$ by $a + b$.

11. $x^5 + x^4y + xy^4 + y^5$ by $x + y$.

12. $a^3 + b^3$ by $a + b$, and $a^5 + b^5$ by $a + b$.

13. $m^3 - n^3$ by $m^2 + mn + n^2$.

14. $a^3 + 4a^2y + 4ay^2 + y^3$ by $a + y$.

15. $a^3 - b^3$ by $a^2 + ab + b^2$.

16. $a^6 + b^6$ by $a^4 - a^2b^2 + b^4$.

17. $81a^4 - 16b^4$ by $3a - 2b$.

18. $x^7 + 1$ by $x + 1$.

19. $16a^4 + 4a^2x^2 + x^4$ by $4a^2 + 2ax + x^2$.

20. $12x^3 + 17x^2y + 26xy^2 + 15y^3$ by $4x + 3y$.

21. $15m^4 - 19m^2n + 5m^2 + 6n^2 - 3n$ by $5m^2 - 3n$.

22. $24x^2 + 28xy + 8xz + 8y^2 + 4yz$ by $8x + 4y$.

23. $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$ by $a^3 - 3a^2b + 3ab^2 - b^3$.

24. $a^4 - 9a^2b^2 + 24ab^3 - 16b^4$ by $a^2 + 3ab - 4b^2$.

25. $a^4 + 4b^4$ by $a^2 + 2ab + 2b^2$.

26. $m^5 + n^5$ by $m + n$.

27. $x^7 - y^7$ by $x - y$.

28. $m^{10} + n^{10}$ by $m^2 + n^2$.

29. $a^{2n} + 2a^nb^n + b^{2n}$ by $a^n + b^n$.

30. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

31. $x^n + y^n$ by $x + y$ to five terms.

32. $a^{4n} + a^{2n}y^{2n} + y^{4n}$ by $a^{2n} - a^ny^n + y^{2n}$.

33. $x^n - 1$ by $x - 1$ to five terms.

34. $x^{5n} - y^{5n}$ by $x^n - y^n$.

35. $a^{12} + b^{12}$ by $a^4 + b^4$.

36. $27a^3 + b^3$ by $9a^2 - 3ab + b^2$.

37. $x^6 + x^4 - x^3 + x^2 + 1$ by $x^2 + x + 1$.

38. $x^6 + 10x - 33$ by $3 + x^2 - 2x$.

39. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

40. $m^4 + m^5 - 2m - 2m^2$ by $m^2 + m$.

41. $13x + 15x^3 - 17x^2 - 3$ by $5x^2 - 4x + 3$.

42. $x^2 + x^3 + 7x^4 + 5x^5$ by $x - x^2$.

43. $x^6 - y^6$ by $x^3 + 2x^2y + 2xy^2 + y^3$.

44. $x^5 + y^5$ by $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

45. $1 - 9y^8 - 8y^9$ by $1 + 2y + y^2$.

SPECIAL METHOD OF DIVIDING A BINOMIAL BY A BINOMIAL.

83. The **Division of a Binomial** by a binomial is greatly abbreviated by the following method:

1. Divide $a^3 + b^3$ and $a^5 + b^5$ by $a + b$.

By actual division,

$$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2.$$

And
$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

Expand the following:

54. $(m - a)(m + a)$.
55. $(m - a)(m + a)(m^2 + a^2)$.
56. $(m + a)(m^2 + a^2)(m - a)(m^4 + a^4)$.
57. $(n^2 - np + p^2)(n^2 + np + p^2)$.
58. $(x + y)(x - y)(x + y)(x - y)$.
59. $(a^2 + a + 1)(a^2 - a + 1)(a + 1)(a - 1)$.
60. $(a^4 - a^2y^2 + y^4)(a^2 + y^2)$.
61. $(x^2 - x + 1)(x^2 + x + 1)(x^4 - x^2 + 1)$.

EQUATIONS AND PROBLEMS.

74. A **Numerical Equation** is one in which the known quantities are expressed by numbers; as,

$$3x + 4 = 2x + 12.$$

75. A **Literal Equation** is one in which some or all of the known quantities are expressed by letters; as,

$$4x - 2x = a + 4.$$

Or, $$4x + 2x = a + b.$$

76. An **Identical Equation** is one in which the two members are the same, or will become the same when the operations indicated are performed; as,

$$x + z = x + z.$$

Or, $$3(x + 5) = 3x + 15.$$

77. The **Root** of an equation is the value of the unknown quantity.

The root is said to be *verified*, and the equation to be *satisfied*, when the value of the root is substituted for the unknown quantity and the equation is reduced to identity.

Thus, in the equation $2x + 6 = 16$, 5 is the root. If 5 be substituted for x, the equation will become $10 + 6 = 16$, an identical equation. Here the root is verified, and the equation is satisfied.

78. Solve the following equations, and verify the result:

1. $5(x+b)=3(x+a)$.

SOLUTION.

$$5(x+b)=3(x+a).$$

Expanding, $5x+5b=3x+3a.$

Transposing, $2x=3a-5b.$

Dividing by 2, $x=\frac{3a-5b}{2}$, *Ans.*

2. $5(x-2)=30$.
3. $4(3x+5)=80$.
4. $3(2x-4+x)=69$.
5. $4x=48-3(x+2)$.
6. $5x-2(2x-8)=12$.
7. $4(x-3)=3(x-2)+2$.
8. $4(2x-5)=2(3x-7)+10$.
9. $3(x-a)=4(x+n)$.
10. $3(x+2)=x+5$.
11. $4(x-3)=2(x+b)$.
12. $(x-2)(x+2)=x^2-x+12$.
13. $4(x+12)+3(2x-9)=61$.
14. $3x+5(x+8)-12=x+56$.
15. $(x+2)(x-3)=x^2-4x+12$.
16. $4(x-3)+5=1-10(2x-4)$.
17. $(x+4)(x-5)+14=(x+8)(x-7)+30$.
18. $(2x+8)(3x-14)=(6x+2)(x-4)+40$.
19. $(x-2)(x+3)-(x+4)(x-5)=x+17$.
20. $(4x-2)(4x-2)-(8x-5)(2x+7)=-209$.

PROBLEMS.

Solve the following problems, and verify the result:

21. The sum of two numbers is 105; what are the numbers if the first is nine times the second, increased by 5?

22. A, B, and C own 350 acres of land: if B had 10 acres more, A would have two times, and C three times, as much as B would then have; how many acres has each?

23. Eight men hire a coach to ride to Lancaster; but before starting two of them decline going, by which the expense of each is increased \$2; what do they pay for the coach?

Solution.

Let $x =$ what each paid by the first condition.

Then $8x =$ the cost of the coach.

Let $x + \$2 =$ what each one paid by the second condition.

Then $6(x + \$2) =$ the cost of the coach.

Then $8x = 6(x + \$2)$.

Or $8x = 6x + \$12$.

And $2x = \$12$.

$x = \$6$.

Whence $8x = \$48$, the cost of the coach.

24. Twelve men hire a boat for sailing; but before starting five of the company decline going, by which the expense of each is increased \$5; what do they pay for the boat?

25. A company of twenty persons engage a dinner; but before paying the bill eight of the company withdraw, by which each person's bill is increased \$4; what do they pay for the dinner?

26. Thirty persons chartered a car for \$600 to visit the World's Fair; but by adding a certain number of persons to the company, each person's expenses were diminished \$5; how many persons were added?

27. A man gave some beggars 6 cents each, and had 12 cents remaining: if he had given them 8 cents, he would have had no money remaining; how many beggars were there?

28. If a square field were 5 feet longer, it would contain 425 square feet more surface; what is its length?

29. B gave some poor families \$5 each, and had \$24 remaining: if he had given them \$7 apiece, he would have had \$16 remaining; how many families were there?

DIVISION.

79. Division is the process of finding how many times one quantity is contained in another.

The **Dividend** is the quantity to be divided.

The **Divisor** is the quantity by which we divide.

The **Quotient** is the result of the process.

PRINCIPLE I.

80. *Taking a factor out of a quantity divides the quantity by that factor.*

If x is taken out of $3xy$ we have $3y$, which is $3xy$ divided by x, since $3y$ multiplied by x equals $3xy$.

PRINCIPLE II.

The exponent of a factor in the quotient equals its exponent in the dividend minus its exponent in the divisor.

Thus, $\frac{a^5}{a^3} = \frac{aaaaa}{aaa} = aa = a^2$ (Prin. 1).

Hence $a^5 \div a^3 = a^{5-3} = a^2$.

Also, $a^{10} \div a^4 = a^{10-4} = a^6$.

And, in general, $a^m \div a^n = a^{m-n}$.

PRINCIPLE III.

The quotient is positive when the dividend and divisor have like signs, and negative when they have unlike signs; or, in general, *the quotient of* **like** *signs gives* **plus,** *and of* **unlike** *signs,* **minus.**

Thus, $+ab \div b = +a$, since $+a \times +b = +ab$.

$-ab \div b = -a$, since $-a \times +b = -ab$.

$+ab \div -b = -a$, since $-a \times -b = +ab$.

$-ab \div -b = +a$, since $+a \times -b = -ab$.

PRINCIPLE IV.

Any quantity whose exponent is zero is equal to 1.

Thus, $\frac{a^4}{a^4} = a^{4-4} = a^0$ (Prin. 2).

$\frac{a^4}{a^4} = 1$. Hence $a^0 = 1$ (Art. 54, Ax. 9).

PRINCIPLE V.

Any quantity, with either a negative or a positive exponent, is equal to the reciprocal of itself with the sign of its exponent changed.

Thus, $\frac{a^4}{a^6} = a^{4-6} = a^{-2}$ (Prin. 2); and $\frac{a^4}{a^6} = \frac{1}{a^2}$, dividing by a^4.

Hence $a^{-2} = \frac{1}{a^2}$ (ART. 54, Ax. 9).

Also, $\frac{a^6}{a^4} = a^{6-4} = a^2$ (Prin. 2); and $\frac{a^6}{a^4} = \frac{1}{a^{-2}}$, dividing by a^6.

Hence $a^2 = \frac{1}{a^{-2}}$ (ART. 54, Ax. 9).

PRINCIPLE VI.

Dividing each term of a quantity divides the quantity.

Since $(a + b)c = ac + bc$, it follows that $(ac + bc) \div c = a + b$.

81. To divide when the divisor is a monomial.

1. Divide $6a^5$ by $2a^3$.

PROCESS.

$$2a^3\overline{)6a^5}$$
$$3a^2, \textit{Ans.}$$

To divide $6a^5$ by $2a^3$, we must take out the factors 2 and a^3 (Prin. 1).

6 divided by 2 equals 3, and a^5 divided by a^3 equals a^2 (Prin. 2).

Hence $6a^5$ divided by $2a^3$ equals $3a^2$.

2. Divide $9a^5b^2 - 6a^4b^5$ by $3a^2b^2$.

PROCESS.

$$3a^2b^2\overline{)9a^5b^2 - 6a^4b^5}$$
$$3a^3 \quad - 2a^2b^3, \textit{Ans.}$$

Since dividing each term of a quantity divides the quantity (Prin. 6), we divide each term of $9a^5b^2 - 6a^4b^5$ by $3a^2b^2$, and obtain $3a^3 - 2a^2b^3$.

From these examples we derive the following

RULE.

Divide each term of the dividend by the divisor. To the quotient of the coefficients annex the literal quantities, giving each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor. Make the quotient + when the dividend and divisor have like signs, and − when they have unlike signs.

EXAMPLES.

3. $2\overline{)4}$
4. $2\overline{)-4}$
5. $-2\overline{)4}$
6. $-2\overline{)-4}$
7. $a\overline{)a^2}$

8. $x^2\overline{)x^4}$
9. $-x^6\overline{)x^9}$
10. $x^4\overline{)-x^7}$
11. $-a^8\overline{)a^{14}}$
12. $-n^4\overline{)-n^{10}}$

13. $a^n\overline{)-a^{2n}}$
14. $-x^{2n}\overline{)-x^{6n}}$
15. $-a^{5n}\overline{)a^{8n}}$
16. $x^2\overline{)-x^{5n}}$
17. $-x^n\overline{)x^8}$

18. $-m^2\overline{)m^n}$
19. $-a^{n-2}\overline{)a^{n+3}}$
20. $a^{m+2}\overline{)a^{m-3}}$
21. $-x^4\overline{)-x^n}$
22. $y^{n+1}\overline{)y^{n-1}}$

Divide—

23. $8a^2x^4$ by $4ax^2$.
24. $12n^2x^3$ by $-6n^2x$.
25. $-9x^4y^4$ by $-3x^2y^3$.
26. $-10y^4a^8$ by $5y^4a^6$.
27. $15a^2b^3c^4$ by $3ab^2c$.
28. $-8a^8b^{12}$ by $-4a^4b^8$.
29. $25m^2n^{10}$ by $5mn^5$.
30. $18x^2y^3b$ by $-6xy^3$.
31. $-15n^4x^3y^2$ by $5n^4x$.
32. $12x^7y^3z^2$ by $-6xz^2$.
33. $-a^n$ by a^m.
34. a^{m+n} by a^{2n}.
35. x^{m-n} by x^{n-m}.
36. x^{5m-4} by $-x^{6-m}$.
37. $12x^{m+n}$ by $-2x^{m-n}$.
38. $a^{3m}b^{2n}$ by $-a^4b^{3m}$.
39. $(a+b)^5$ by $(a+b)^3$.
40. $(a+b)^n$ by $(a+b)^m$.
41. $(x+y)^{m+n}$ by $(x+y)^{m-n}$.
42. $(a-b)^{x-y}$ by $(a-b)^{x+y}$.
43. $8a^2b^3-12ab^2$ by $4ab$.
44. $15a^3n^2-20a^2m^3$ by $-5a^2$.
45. $42x^3y^2-18x^2y^3$ by $6x^2y^2$.
46. $8m^4n-6mn^4$ by $-2mn$.
47. $7x^5-5x^4-4x^3$ by $-x^2$.
48. $34x^6y^4-51x^4y^6$ by $17x^4y^4$.
49. $100x^2-50x$ by $25x$.
50. $x^2+2xy+yx^2$ by $-x$.
51. $m^2n+mn^2+n^3$ by $-n$.
52. $12x^2n^4-6x^4n$ by $3x^2n^2$.
53. $16a^4m^8+10a^5m^9$ by $\frac{1}{2}a^4m^9$.
54. $\frac{1}{2}a^4y^6-6a^6by^4$ by $-\frac{1}{3}a^3y^3$.
55. $6m^{10}n^8-8m^{11}n^{12}$ by $\frac{1}{2}m^4n^6$.
56. $50m^4x^9-9m^4x^5$ by $-\frac{1}{3}m^4x^3$.

57. $18a^2b^4c^3 - 24a^4b^3c - 30a^2b^3c^4$ by $6a^2b^2c$.

58. $x^2y^2 + 2xy^3 + 3xy^2 - 4x^3y^2$ by $-xy$.

59. $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ by x^3.

60. $6a^{4n} - 12a^{3n} + 18a^{2n} - 24a^n + 30$ by $6a^{2n}$.

61. $y^{n+4} - y^{n+3} + y^{n+2} + y^{n+1} + y$ by y.

62. $x^{n+1} + x^{n+2} + x^{n+3} + x^{n+4}$ by x^3.

63. $6(a+b) - 18(a+b)^2 + 12(a+b)^4$ by $-3(a+b)^2$.

64. $3a(x+y)^n + 6a(x+y)^{2n} - a(x+y)^{3n}$ by $a(x+y)^n$.

82. To divide when the divisor is a polynomial.

1. Divide $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.

PROCESS.

$$\begin{array}{r|l}
 & \underline{a + b,\ \text{Divisor.}} \\
a^3 + 3a^2b + 3ab^2 + b^3 & \underline{a^2 + 2ab + b^2},\ \text{Quotient.} \\
\underline{a^3 + a^2b \phantom{{}+3ab^2+b^3}} & \\
2a^2b + 3ab^2 \phantom{{}+b^3} & \\
\underline{2a^2b + 2ab^2 \phantom{{}+b^3}} & \\
ab^2 + b^3 & \\
ab^2 + b^3 &
\end{array}$$

For convenience of multiplying, write the divisor at the right of the dividend, and arrange both dividend and divisor according to the descending (or ascending) powers of a.

Since the first term of the dividend, a^3, is equal to the product of the first term of the divisor, a, by the first term of the quotient, we can find the first term of the quotient by dividing a^3 by a. a is contained in a^3, a^2 times; a^2 times $a + b$ equals $a^3 + a^2b$. Subtracting, and bringing down the next term, we have $2a^2b + 3ab^2$.

Since the first term of this new dividend, $2a^2b$, is equal to the product of the first term of the divisor, a, by the second term of the quotient, we can find the second term of the quotient by dividing $2a^2b$ by a. a is contained in $2a^2b$, $2ab$ times; $2ab$ times $a + b$ equals $2a^2b + 2ab^2$. Subtracting, and bringing down the next term, we have $ab^2 + b^3$.

Since the first term of this new dividend, ab^2, is equal to the product of the first term of the divisor, a, by the third term of the quotient, we can find the third term of the quotient by dividing ab^2 by a. a is contained in ab^2, b^2 times; b^2 times $a + b$ equals $ab^2 + b^3$. Subtracting, there is no remainder. Hence the quotient is $a^2 + 2ab + b^2$.

From this example we derive the following

RULE.

Arrange both dividend and divisor according to the ascending or descending powers of a common letter.

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Multiply the divisor by the first term of the quotient, and subtract the product from the dividend.

Treat the remainder as a new dividend, and proceed as before.

Continue the process until the remainder does not contain the leading letter.

If there is a remainder after the last division, write it over the denominator and annex it with its proper sign to the quotient.

EXAMPLES.

2.
$$\begin{array}{l|l}
a^3 - b^3 & \underline{a - b} \\
\underline{a^3 - a^2b} & a^2 + ab + b^2 \\
\quad a^2b - b^3 & \\
\quad \underline{a^2b - ab^2} & \\
\qquad ab^2 - b^3 & \\
\qquad \underline{ab^2 - b^3} &
\end{array}$$

3.
$$\begin{array}{l|l}
x^4 - a^4 & \underline{x - a} \\
\underline{x^4 - x^3a} & x^3 + x^2a + xa^2 + a^3 \\
\quad x^3a - a^4 & \\
\quad \underline{x^3a - x^2a^2} & \\
\qquad x^2a^2 - a^4 & \\
\qquad \underline{x^2a^2 - xa^3} & \\
\qquad\quad xa^3 - a^4 & \\
\qquad\quad \underline{xa^3 - a^4} &
\end{array}$$

4.
$$\begin{array}{l|l}
x^4 + x^2y^2 + y^4 & \underline{x^2 + xy + y^2} \\
\underline{x^4 + x^3y + x^2y^2} & x^2 - xy + y^2 \\
\quad - x^3y + y^4 & \\
\quad \underline{- x^3y - x^2y^2 - xy^3} & \\
\qquad x^2y^2 + xy^3 + y^4 & \\
\qquad \underline{x^2y^2 + xy^3 + y^4} &
\end{array}$$

$$\begin{array}{lll}
5. & a^n - b^n & \left| \; a - b \right. \\ \cline{3-3}
 & \underline{a^n - a^{n-1}b} & \left| \; \underline{a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \text{etc.}} \right. \\
 & \quad a^{n-1}b - b^n & \\
 & \quad \underline{a^{n-1}b - a^{n-2}b^2} & \\
 & \qquad\qquad a^{n-2}b^2 - b^n & \\
 & \qquad\qquad \underline{a^{n-2}b^2 - a^{n-3}b^3} & \\
 & \qquad\qquad\qquad\qquad a^{n-3}b^3 - b^n, \text{ etc.} &
\end{array}$$

Divide—

6. $x^2 + 2xy + y^2$ by $x + y$.

7. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.

8. $a^2 - 6ax + 9x^2$ by $a - 3x$.

9. $16x^4 - 8x^2a^2 + a^4$ by $4x^2 - a^2$.

10. $a^3 + a^2b + ab^2 + b^3$ by $a + b$.

11. $x^5 + x^4y + xy^4 + y^5$ by $x + y$.

12. $a^3 + b^3$ by $a + b$, and $a^5 + b^5$ by $a + b$.

13. $m^3 - n^3$ by $m^2 + mn + n^2$.

14. $a^3 + 4a^2y + 4ay^2 + y^3$ by $a + y$.

15. $a^3 - b^3$ by $a^2 + ab + b^2$.

16. $a^6 + b^6$ by $a^4 - a^2b^2 + b^4$.

17. $81a^4 - 16b^4$ by $3a - 2b$.

18. $x^7 + 1$ by $x + 1$.

19. $16a^4 + 4a^2x^2 + x^4$ by $4a^2 + 2ax + x^2$.

20. $12x^3 + 17x^2y + 26xy^2 + 15y^3$ by $4x + 3y$.

21. $15m^4 - 19m^2n + 5m^2 + 6n^2 - 3n$ by $5m^2 - 3n$.

22. $24x^2 + 28xy + 8xz + 8y^2 + 4yz$ by $8x + 4y$.

23. $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$ by $a^3 - 3a^2b + 3ab^2 - b^3$.

24. $a^4 - 9a^2b^2 + 24ab^3 - 16b^4$ by $a^2 + 3ab - 4b^2$.

25. $a^4 + 4b^4$ by $a^2 + 2ab + 2b^2$.

26. $m^5 + n^5$ by $m + n$.

27. $x^7 - y^7$ by $x - y$.

28. $m^{10} + n^{10}$ by $m^2 + n^2$.

29. $a^{2n} + 2a^nb^n + b^{2n}$ by $a^n + b^n$.

30. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

31. $x^n + y^n$ by $x + y$ to five terms.

32. $a^{4n} + a^{2n}y^{2n} + y^{4n}$ by $a^{2n} - a^ny^n + y^{2n}$.

33. $x^n - 1$ by $x - 1$ to five terms.

34. $x^{5n} - y^{5n}$ by $x^n - y^n$.

35. $a^{12} + b^{12}$ by $a^4 + b^4$.

36. $27a^3 + b^3$ by $9a^2 - 3ab + b^2$.

37. $x^6 + x^4 - x^3 + x^2 + 1$ by $x^2 + x + 1$.

38. $x^6 + 10x - 33$ by $3 + x^2 - 2x$.

39. $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

40. $m^4 + m^5 - 2m - 2m^2$ by $m^2 + m$.

41. $13x + 15x^3 - 17x^2 - 3$ by $5x^2 - 4x + 3$.

42. $x^2 + x^3 + 7x^4 + 5x^5$ by $x - x^2$.

43. $x^6 - y^6$ by $x^3 + 2x^2y + 2xy^2 + y^3$.

44. $x^5 + y^5$ by $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

45. $1 - 9y^8 - 8y^9$ by $1 + 2y + y^2$.

SPECIAL METHOD OF DIVIDING A BINOMIAL BY A BINOMIAL.

83. The **Division of a Binomial** by a binomial is greatly abbreviated by the following method:

1. Divide $a^3 + b^3$ and $a^5 + b^5$ by $a + b$.

By actual division,

$$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2.$$

And
$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

2. Divide $a^3 - b^3$ and $a^4 - b^4$ by $a - b$.

By actual division,

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2.$$

And $$\frac{a^4 - b^4}{a - b} = a^3 + a^2b + ab^2 + b^3.$$

By examining these results we perceive that—

1. *When the divisor is* $a + b$, *the signs of the quotient are alternately plus and minus.*

2. *When the divisor is* $a - b$, *the signs of the quotient are all positive.*

3. *The first term of the quotient is found by dividing the first term of the dividend by the first term of the divisor.*

4. *Any succeeding term of the quotient may be found by dividing the term of the quotient last found by the first term of the divisor, and multiplying the result by the second term of the divisor.*

3. Divide $a^6 + b^6$ by $a^2 + b^2$.

$$\frac{a^6 + b^6}{a^2 + b^2} = a^4 - a^2b^2 + b^4.$$

Since the divisor is $a^2 + b^2$, the signs of the quotient are alternately plus and minus.

The first term of the quotient is $a^6 \div a^2 = a^4$.
The second term is $(a^4 \div a^2)b^2 = a^2b^2$.
The third term is $(a^2b^2 \div a^2)b^2 = b^4$.

4. Divide $8x^3 - 27y^3$ by $2x - 3y$.

$$\frac{8x^3 - 27y^3}{2x - 3y} = 4x^2 + 6xy + 9y^2.$$

5. Divide $a^{10} + b^{10}$ by $a^2 + b^2$.

$$\frac{a^{10} + b^{10}}{a^2 + b^2} = a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8.$$

EXAMPLES.

Divide the following binomials:

6. $\frac{a^2-b^2}{a-b}$.
7. $\frac{a^2-b^2}{a+b}$.
8. $\frac{a^3+b^3}{a+b}$.
9. $\frac{a^3-b^3}{a-b}$.
10. $\frac{a^4-b^4}{a+b}$.
11. $\frac{a^4-b^4}{a-b}$.
12. $\frac{a^5+b^5}{a+b}$.
13. $\frac{a^5-b^5}{a-b}$.
14. $\frac{a^6-b^6}{a+b}$.
15. $\frac{a^6-b^6}{a^2-b^2}$.
16. $\frac{a^6+b^6}{a^2+b^2}$.
17. $\frac{a^7-b^7}{a-b}$.
18. $\frac{a^7+b^7}{a+b}$.
19. $\frac{a^8-b^8}{a+b}$.
20. $\frac{a^8-b^8}{a-b}$.
21. $\frac{x^9-y^9}{x-y}$.
22. $\frac{x^9+y^9}{x+y}$.
23. $\frac{x^9+y^9}{x^3+y^3}$.
24. $\frac{x^9-y^9}{x^3-y^3}$.
25. $\frac{x^{10}-y^{10}}{x-y}$.
26. $\frac{x^{10}-y^{10}}{x+y}$.
27. $\frac{x^{10}+y^{10}}{x^2+y^2}$.
28. $\frac{a^{12}+x^{12}}{a^4+x^4}$.
29. $\frac{x^{14}+y^{14}}{x^2+y^2}$.
30. $\frac{x^6+y^9}{x^2+y^3}$.
31. $\frac{a^{12}+b^3}{a^4+b}$.
32. $\frac{m^{30}-n^5}{m^6-n}$.
33. $\frac{1-x^8}{1-x}$.
34. $\frac{64x^3-27y^3}{4x-3y}$.
35. $\frac{x^6+729a^3}{x^2+9a}$.
36. $\frac{a^3-1}{a-1}$.
37. $\frac{m^3-125}{m-5}$.
38. $\frac{81x^4-16}{3x+2}$.
39. $\frac{a^7-1}{a-1}$.
40. $\frac{1-n^9}{1-n}$.
41. $\frac{729x^6+y^6}{9x^2+y^2}$.
42. $\frac{243a^5-b^5}{3a-b}$.
43. $\frac{27a^3b^3+8c^3}{3ab+2c}$.
44. $\frac{256x^4-a^4}{4x-a}$.
45. $\frac{625m^4-n^4}{5m+n}$.
46. $\frac{512x^3-8y^3}{8x-2y}$.
47. $\frac{32x^5-243y^5}{2x-3y}$.
48. $\frac{1-x^n}{1-x}$.
49. $\frac{x^n-a^n}{x+a}$.
50. $\frac{x^n-a^n}{x-a}$.

REVIEW.

84.—1. Add $2a + 3b - 4c$, $5a - 7b + 5c$, $-3a - 4b - 8c$, $10a + 4b + 6c$, $11a - 14b - c$, and $11a - 5c + 2b$.

2. Add $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$, $3a^2 - 6ab + 3b^2$, $4a^2 + 4ab - 4b^2$, $5a^2 + 6ab + 4b^2$, $8a^2 - 4ab + b^2$, and $10a^2 - 8ab - 6b^2$.

3. Subtract $8m^2 - 6mn - 5n^2$ from $12m^2 + 10mn - 6n^2$.

4. Subtract $19x^2 + 12xy - 6y^2$ from $25x^2 - 14y^2 - 8xy$.

5. Multiply $a^2 + 2ab + b^2$ by $a + b$.

6. Multiply $m^2 - n^2$ by $m^2 + n^2$.

7. Divide $a^4 - b^6$ by $a^2 + b^3$.

8. Divide $a^{10} - b^5$ by $a^2 - b$.

9. Solve $8x - 6 + 5x - 9 = 4x + 30$.

10. Solve $11x - 14 - 5x = 18 - 7x + 7$.

11. Divide $x^7 - y^{14}$ by $x - y^2$.

12. Divide $x^5y - xy^5$ by $x^2y - xy^2$.

13. Multiply $(a - b)(a + b)(a^2 + b^2)$.

14. Add $am^2 - bn^3 - 14mn$, $-cm^2 + dn^3 + 12mn$, $bm^2 + an^3 + 9mn$, and $(c - b - a)m^2 - (d - b + a)n^3 + mn$.

15. From $a(a - x) + b\sqrt{a^2 - x^2} + 4$ take $a\sqrt{a^2 - x^2} - 3(a - x) - 5$.

16. Multiply $x^2 - xy + y^2$ by $x^2 + xy + y^2$.

17. Divide $4x^4 - 5x^2y^2 + y^4$ by $2x^2 - 3xy + y^2$.

18. Find the value of $a + 3(a - b)^2 - a + \{4b - 2(b - a)^2\}$.

19. Expand $(a - b + c)(a - b - c)$.

20. Add $a^4 - 2a^3 + 3a^2$, $a^3 + a^2 + a$, $4a^4 + 5a^3$, $2a^2 + 3a - 4$, and $-3a^2 - 2a - 5$.

21. From $8\sqrt{x} - 3\sqrt{y} - 2\sqrt{x + y}$ take $5\sqrt{x} - a\sqrt{y} + 5\sqrt{x + y}$.

22. Multiply $(m - a + b)$ by $(m + a + b)$.

23. Divide $4x^4 + y^4 - 5x^2y^2$ by $2x^2 + y^2 + 3xy$.

24. Find the value of $-\{a + b - [a - (a - \overline{b - c}) + b] - b + c\}$.

25. Expand $(x^{n^2} - y^n)(x^{n^2} + y^n)$.

26. Divide $x^7 - y^7$ by $x - y$.

27. Add $5xy - 10 + 3\sqrt{x}$, $10xy + 4 - 8\sqrt{x}$, $8 - 12xy + 9\sqrt{x} + a$, $\sqrt{x} - 12 - 8xy - 2a$, and $10 - 5\sqrt{xy}$.

28. From $2\sqrt{x} + xy - \sqrt{y}$ take $\sqrt{x} - 5xy - a\sqrt{y}$.

29. Find the value of $4a - [2a - (6a - \overline{10a - 2a})]$.

30. Multiply $x^n - xy + y^n$ by $x^n + xy + y^n$.

31. Divide $m^5 - 6m^3 - 7m^2 - 6m$ by $m^3 + m^2 + m$.

32. Add $ax + by + cz$, $mx - my - pz$, $-2ax - by + (c + p)z$, and $(n + a)x - (m - a)y - 2cz$.

33. From $a^n - a^{n-1} + a^{n-2} - a^{n-3} + a^{n-4}$ take $a^{n-2} - a^{n-3} + a^{n-4} - a$.

34. Multiply $x^x + y^y$ by $x^x - y^y$.

35. Divide $a^{4n} + a^{2n}y^{2n} + y^{4n}$ by $a^{2n} + a^n y^n + y^{2n}$.

36. Find the value of $x - \{3x + 5x - [8x - (3x - 6x)]\}$.

37. Expand $(3x + 5)(3x - 5)(9x^2 + 25)$.

38. Divide $a^{24} + b^{24}$ by $a^8 + b^8$.

39. Multiply $(a - 3)(a - 3)$ by $(a + 3)(a + 3)$.

40. Divide $a^8 + 12a^6x + 54a^4x^2 + 108a^2x^3 + 81x^4$ by $a^4 + 9x^2 + 6a^2x$.

41. Find the value of $2x - \{3a + (2y - \overline{4a - 6x})\} - \{3a + (5x - \overline{8a - 5y})\}$.

42. Expand $(x^3 + y^3)^2$ and $(a^5 - n^5)^3$.

43. Divide $a^{6x} + b^{6y}$ by $a^{2x} + b^{2y}$.

44. Multiply $x^4 + x^2 + 1$ by $x^4 - x^2 + 1$.

45. Divide $x^{3n} - y^{3n}$ by $x^n - y^n$.

46. Expand $(x^2 + y^2)(x^4 - x^2y^2 + y^4)$.

47. Divide $a^{10} + b^{15}$ by $a^2 + b^3$.

IV. COMPOSITION AND FACTORING.

COMPOSITION.

85. Composition is the process of forming *composite* quantities.

86. A **Composite Quantity** is the product of two or more quantities.

PRINCIPLE I.

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.

Thus, by multiplication,

$$\begin{array}{l} \phantom{a^2+{}}a + b \\ \underline{\phantom{a^2+{}}a + b\phantom{{}+b^2}} \\ a^2 + ab \\ \underline{+ ab + b^2} \\ a^2 + 2ab + b^2 \end{array}$$

Also, $(m + n)^2 = m^2 + 2mn + n^2$.

And $(2a + 3b)^2 = 4a^2 + 12ab + 9b^2$.

EXAMPLES.

Write by inspection the values of the following:

1. $(a + x)^2$.
2. $(a + n)^2$.
3. $(c + d)^2$.
4. $(r + s)^2$.
5. $(a + 3)^2$.
6. $(x + 2)^2$.
7. $(5 + y)^2$.
8. $8(a^2 + b^2)^2$.
9. $(2x + 3y)^2$.
10. $(2c + 5d)^2$
11. $(3x + 4)^2$.
12. $(5 + 2x)^2$.
13. $(3x + 5y)^2$.
14. $(a^3 + x^3)^2$.
15. $(2x^2 + a^2)^2$.
16. $(m + 4n)^2$.
17. $(a + 1)^2$.
18. $(a^n + 3b^n)^2$.
19. $(x^{2n} + y^{3n})^2$.
20. $(2x^2 + 3x^3)^2$.
21. $(a^{m+1} + b^{n+1})^2$,
22. $(x^x + y^x)^2$.
23. $(3x^n + 4y^n)^2$.
24. $(x^{n+1} + 2a^{n+1})^2$.
25. $(nx^2 + my^2)^2$.
26. $(ay^3 + by^4)^2$.
27. $(ax^n + by^m)^2$.
28. $(x^{4n} \quad y^{5m})^2$.
29. $(A^2 + B^2)^2$.
30. $(m^{2x} + n^{3y})^2$.

PRINCIPLE II.

The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.

Thus, by multiplication,

$$\begin{array}{l} a - b \\ \underline{a - b} \\ a^2 - ab \\ \underline{\quad - ab + b^2} \\ a^2 - 2ab + b^2 \end{array}$$

Also, $(m - n)^2 = m^2 - 2mn + n^2.$

And $(2a - 3b)^2 = 4a^2 - 12ab + 9b^2.$

EXAMPLES.

Write by inspection the values of the following:

1. $(a - x)^2$.
2. $(m - a)^2$.
3. $(x - 1)^2$.
4. $(b - 2)^2$.
5. $(r - s)^2$.
6. $(p - q)^2$.
7. $(a^2 - y^2)^2$.
8. $(a - 3)^2$.
9. $(x - 4)^2$.
10. $(a^3 - b^3)^2$.
11. $(2x - 3y)^2$.
12. $(3a - 5b)^2$.
13. $(6x - 8y)^2$.
14. $(a - 10y)^2$.
15. $(4p - 6q)^2$.
16. $(2a^2 - 3x^3)^2$.
17. $(x^3 - y^3)^2$.
18. $(4x^2 - 5y^3)^2$.
19. $(ax^2 - by^2)^2$.
20. $(4 - x^2)^2$.
21. $(x^4 - y^4)^2$.
22. $(n^3 - m^3)^2$.
23. $(2x^4 - 3x^2)^2$.
24. $(a^5 - b^5)^2$.
25. $(a^n - b^n)^2$.
26. $(x^{2n} - y^{2n})^2$.
27. $(x^{n+1} - y^{n+1})^2$.
28. $(a^{n-1} - y^{n-1})^2$.
29. $(x^{m+n} - y^{m+n})^2$.
30. $(x^{n-4} - y^{3n-5})^2$.

PRINCIPLE III.

The product of the sum and difference of two quantities equals the difference of their squares.

Thus, by multiplication,

$$\begin{array}{l} a + b \\ \underline{a - b} \\ a^2 + ab \\ \underline{\quad - ab - b^2} \\ a^2 \quad\;\; - \quad b^2 \end{array}$$

Also, $(m + n)(m - n) = m^2 - n^2.$

And $(2a + 3b)(2a - 3b) = 4a^2 - 9b^2.$

EXAMPLES.

Write by inspection the values of the following:

1. $(a+x)(a-x)$.
2. $(m+x)(m-x)$.
3. $(x+y)(x-y)$.
4. $(c+d)(c-d)$.
5. $(x+3)(x-3)$.
6. $(3-x)(3+x)$.
7. $(x-1)(x+1)$.
8. $(a+2b)(a-2b)$.
9. $(a-8x)(a+8x)$.
10. $(3x-7)(3x+7)$.
11. $(a^2-y^2)(a^2+y^2)$.
12. $(3m+4n)(3m-4n)$.
13. $(3a-5c)(3a+5c)$.
14. $(a^n+b^n)(a^n-b^n)$.
15. $(3v^2-4t^2)(3v^2+4t^2)$.
16. $(c^{3n}+d^{2n})(c^{3n}-d^{2n})$.
17. $(ax^n+bx^m)(ax^n-bx^m)$.
18. $(a^{m+n}+b^{m+n})(a^{m+n}-b^{m+n})$.
19. $(a^{n^2}-b^{m^2})(a^{n^2}+b^{m^2})$.
20. $(x^x+y^y)(x^x-y^y)$.

PRINCIPLE IV.

The product of two binomials having a common term equals the square of the common term, plus the algebraic sum of the other two terms into the common term, and the product of the unlike terms.

Thus, by multiplication,

$$\begin{array}{l} x + a \\ \underline{x + b} \\ x^2 + ax \\ \underline{\quad + bx + ab} \\ x^2 + (a+b)x + ab \end{array}$$

Also, $(m+x)(m+y) = m^2 + (x+y)m + xy$.

And $(m+x)(m-y) = m^2 + (x-y)m - xy$.

And $(m-x)(m-y) = m^2 - (x+y)m + xy$.

And $(m+6)(m-2) = m^2 + 4m - 12$.

EXAMPLES.

Write by inspection the values of the following:

1. $(x+1)(x+2)$.
2. $(x+4)(x-1)$.
3. $(x-4)(x+3)$.
4. $(x-2)(x-5)$.
5. $(m+5)(m-8)$.
6. $(x-7)(x+5)$.

7. $(a-8)(a+9)$.
8. $(x-5)(x-10)$.
9. $(a+5)(a-16)$.
10. $(n-10)(n+1)$.
11. $(3x-5)(3x+2)$.
12. $(a+b)(a+e)$.
13. $(x-a)(x+c)$.
14. $(x-b)(x-e)$.
15. $(2x+1)(2x+3)$.
16. $(x^2-4)(x^2-6)$.
17. $(7y-4)(7y-8)$.
18. $(4a-2b)(4a-3c)$.
19. $(a^n-b)(a^n-e)$.
20. $(3x^n-m^n)(3x^n+n^m)$.

GENERAL APPLICATION.

87. These principles may be applied with great convenience in finding the product of three or more binomials or the product of certain polynomials.

1. Expand $(a-b)(a+b)(a^2+b^2)$.

By Prin. III., $(a-b)(a+b)=a^2-b^2$.

Applying Prin. III. again, $(a^2-b^2)(a^2+b^2)=a^4-b^4$.

Hence, $(a-b)(a+b)(a^2+b^2)=a^4-b^4$.

2. Expand $(\overline{a+b}-c)(\overline{a+b}+c)$.

By Prin. III., this equals $(a+b)^2-c^2$.

By Prin. I., $(a+b)^2-c^2=a^2+2ab+b^2-c^2$.

Hence, $(a+b-c)(a+b+c)=a^2+2ab+b^2-c^2$.

Also, $(a-b-c)(a+b+c)=(a-\overline{b+c})(a+\overline{b+c})$
$=a^2-(b+c)^2=a^2-b^2-2bc-c^2$.

EXAMPLES.

Expand—

3. $(1-a)(1+a)(1+a^2)$.
4. $(x-y)(x+y)(x^2+y^2)$.
5. $(a+3)(a-3)(a^2+9)$.
6. $(m+n)(m-n)(m^2-n^2)$.
7. $(a+3)(a+3)(a+3)$.
8. $(a+b)(a+b)(a+b)$.

9. $(x - y - z)(x - y - z)$.

10. $(x^2 - xy - y^2)(x^2 - xy - y^2)$.

11. $(m^2 - mn - n^2)(m^2 - mn - n^2)$.

12. $(x^2 - x - 1)(x^2 - x - 1)$.

13. $(x^2 - xy - y^2)(x^2 - xy - y^2)$.

14. $(1 - a - a^2)(1 - a - a^2)$.

15. $(3x^2 - 3x - 1)(3x^2 - 3x - 1)$.

16. $(m - n)(m - n)(m^2 - n^2)(m^4 - n^4)$.

17. $(1 - a)(1 - a)(1 - a^2)(1 - a^4)(1 - a^8)$.

18. $(a - x)(a - x)(a^2 - x^2)(a^4 - x^4)(a^8 + x^8)$.

19. $(a - b - m - n)(a - b - m - n)$.

20. $(a - m - b - n)(a - m - b - n)$.

21. $(x^n - b^n)(x^n - b^n)(x^{2n} - b^{2n})(x^{4n} - b^{4n})$.

22. $(x^n - y^n - b^n - x^n)(x^n - y^n - b^n - x^n)$.

FACTORING.

88. **Factoring** is the process of resolving a composite quantity into its factors.

The *Factors* of a composite quantity are the quantities which when multiplied together will produce it.

89. A **Prime Quantity** is one that cannot be produced by the multiplication of other quantities: as, 17, a, $a + b$, or $a^2 + b^2$.

In speaking of prime quantities in algebra we must exclude factoring by means of *fractional exponents* and *imaginary quantities*;* otherwise many quantities regarded as prime could be factored.

For example, $a - b = (a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.

Or, $= (a^{\frac{1}{3}} - b^{\frac{1}{3}})(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})$, etc.

$$a^2 + b^2 = (a + b\sqrt{-1})(a - b\sqrt{-1}).$$

---

* An imaginary quantity is the indicated even root of a negative quantity; as, $\sqrt{-a}$, or $\sqrt[2n]{-a}$.

90. By ART. 8, one of the two equal factors of a quantity is its *square root.*

Thus, $2a$ is the square root of $4a^2$.

I. TO FACTOR MONOMIALS.

91. The prime factors of monomials may be obtained by inspection, and therefore need no further consideration.

Thus, the prime factors of $6a^2b^3 = 2 \times 3aabbb$.

II. TO FACTOR BINOMIALS.

92. When the binomial is the difference of two squares.

1. Factor $a^2 - b^2$ and $4x^2 - 9y^2$.

Referring to ART. 86, Prin. III., we see that such binomials are the product of the sum and difference of two quantities.

$$\text{Hence}\quad a^2 - b^2 = (a + b)(a - b).$$
$$4x^2 - 9y^2 = (2x + 3y)(2x - 3y).$$

Therefore, to obtain the factors use the following

RULE.

Extract the square root of each term of the binomial, and make their sum one factor and their difference the other factor.

NOTE.—This is one of the most general and useful cases of factoring, and should always be used first when applicable.

EXAMPLES.

Factor—

2. $m^2 - n^2$.
3. $c^2 - d^2$.
4. $x^2 - y^2$.
5. $p^2 - q^2$.
6. $x^2 - 1$.
7. $4 - y^2$.
8. $x^2 - 4y^2$.
9. $4a^2 - 9b^2$.
10. $16m^2 - 25n^2$.
11. $36x^2 - 49a^2$.
12. $36l^2 - 64m^2$.
13. $x^2y^2 - a^2b^2$.
14. $m^4 - n^4$.
15. $a^8 - b^8$.
16. $a^4 - x^8$.
17. $m^{16} - n^{16}$.
18. $x^{16} - y^8$.
19. $a^{2n} - b^{2m}$.
20. $9a^{2n} - 16b^{2n}$.
21. $a^{14} - b^4$.
22. $x^{8n} - y^{8m}$.

23. $\frac{1}{4}x^2 - \frac{1}{9}y^2$.

24. $225x^2y^2 - 9$.

25. $289x^{2n} - 121y^{2n}$.

26. $81m^2 - 100n^6$.

27. $a^2b^6 - c^2d^8$.

28. $\frac{1}{16}x^4 - 1$.

29. $144 - a^2b^{12}$.

30. $a^{32} - b^{32}$.

31. $a^{16n} - b^{16n}$.

The same principle may be applied to the following binomial forms:

32. Factor $a^2 - (b + c)^2$.

$$\text{1st factor} = [a + (b + c)] = (a + b + c).$$
$$\text{2d factor} = [a - (b + c)] = (a - b - c).$$

33. Factor $(a + b)^2 - (c + d)^2$.

$$\text{1st factor} = [(a + b) + (c + d)] = (a + b + c + d).$$
$$\text{2d factor} = [(a + b) - (c + d)] = (a + b - c - d).$$

34. $(a + b)^2 - c^2$.

35. $(m + n)^2 - p^2$.

36. $(x + y)^2 - z^2$.

37. $m^2 - (x - y)^2$.

38. $(a - b)^2 - c^2$.

39. $a^2 - (c + x)^2$.

40. $4 - (a - b)^2$.

41. $(x - y)^2 - 9$.

42. $m^4 - (m^2 - 1)^2$.

43. $x^2 - (x + y)^2$.

44. $(3x - 5)^2 - 4x^2$.

45. $m^6 - (m^3 + 1)^2$.

46. $(x + y)^2 - (z + v)^2$.

47. $(m + n)^2 - (x - y)^2$.

48. $(a - x)^2 - (y + b)^2$.

49. $(m - n)^2 - (p - q)^2$.

50. $(x + y)^2 - (x - y)^2$.

51. $(a - b)^2 - (a + b)^2$.

52. $(2a - c)^2 - (3b - x)^2$.

53. $(2x - 4)^2 - (x - 4)^2$.

54. $(5x + 2y)^2 - (5 - 3y)^2$.

55. $(x - 1)^2 - (y + 1)^2$.

56. $(ax + by)^2 - (ax - by)^2$.

57. $(m^2 + n^2)^2 - (a^2 - b^2)^2$.

Polynomials can sometimes be arranged in the form of the difference of two squares, and factored like the preceding examples.

58. Factor $x^2 + 2xy + y^2 - z^2$.

This may be arranged and factored as follows:

$$x^2 + 2xy + y^2 - z^2 = (x+y)^2 - z^2 = (x + y + z)(x + y - z).$$

59. Factor $a^2 + b^2 - c^2 - d^2 - 2cd - 2ab$.

This may be factored thus:

$$\begin{aligned}
&a^2 + b^2 - c^2 - d^2 - 2cd - 2ab \\
&= a^2 - 2ab + b^2 - c^2 - 2cd - d^2 \\
&= (a^2 - 2ab + b^2) - (c^2 + 2cd + d^2) \\
&= (a - b)^2 - (c + d)^2 \\
&= [(a - b) + (c + d)]\,[(a - b) - (c + d)] \\
&= (a - b + c + d)\,(a - b - c - d).
\end{aligned}$$

60. $a^2 + 2ab + b^2 - c^2$.
61. $x^2 - 2xy + y^2 - z^2$.
62. $x^2 - a^2 - 2ab - b^2$.
63. $y^2 - m^2 + 2mn - n^2$.
64. $1 - m^2 - n^2 - 2mn$.
65. $2ab - a^2 - b^2 + 1$.
66. $a^2 - 2xy - x^2 - y^2$.
67. $a^2 - y^2 + b^2 - 2ab$.
68. $x^2 - 2yz + y^2 - z^2$.
69. $m^2 - 2mn + n^2 - p^2$.
70. $m^2 - n^2 + 2pn - p^2$.
71. $m^2 - n^2 - 2pn - p^2$.
72. $m^2 + n^2 - p^2 + 2mn$.
73. $x^2 - 1 + y^2 + 2xy$.
74. $a^2 + 2ab + b^2 - x^2 + 2xy - y^2$.
75. $m^2 - 2mn + n^2 - p^2 - 2pq - q^2$.
76. $a^2 - y^2 - b^2 + x^2 - 2ax + 2by$.
77. $a^2 - 9y^2 - 6xy + 4b^2 - 4ab - x^2$.
78. $a^2 - 9y^2 + 9x^2 - 4z^2 + 12yz + 6ax$.
79. $m^2 - 2xy + n^2 - x^2 + 2mn - y^2$.

Sometimes a quantity may be both *added to*, and *subtracted from*, a polynomial, and the terms then arranged in the form of the difference of two squares and factored like the preceding examples.

80. Factor $x^4 + x^2y^2 + y^4$.

Since $x^4 + x^2y^2 + y^4$ lacks x^2y^2 of being a perfect square, if x^2y^2 be added *to* the expression and then subtracted *from* it, the value will not be changed (ART. 54, Axiom 5).

Adding and subtracting x^2y^2, the expression becomes

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2 \\ &= [(x^2 + y^2) + xy]\,[(x^2 + y^2) - xy] \\ &= (x^2 + xy + y^2)\,(x^2 - xy + y^2). \end{aligned}$$

81. Factor $a^4 + a^2 + 1$.

Adding and subtracting a^2,

$$\begin{aligned} a^4 + a^2 + 1 &= a^4 + 2a^2 + 1 - a^2 \\ &= (a^2 + 1)^2 - a^2 \\ &= (a^2 + a + 1)\,(a^2 - a + 1). \end{aligned}$$

82. $m^4 + m^2n^2 + n^4$.

83. $1 + y^2 + y^4$.

84. $b^4 + b^2c^2 + c^4$.

85. $a^4 + a^2x^2 + x^4$.

86. $x^4 - 3x^2y^2 + y^4$.

87. $a^4 - 6a^2n^2 + n^4$.

88. $a^4 - 7a^2y^2 + y^4$.

89. $m^4 - 27m^2n^2 + n^4$.

90. $l^4 - 38l^2m^2 + m^4$.

91. $4x^4 - 16x^2y^2 + 9y^4$.

92. $9a^4 + 14a^2b^2 + 25b^4$.

93. $25a^4 + a^2x^2 + x^4$.

94. $25x^4 - 79x^2y^2 + 49y^4$.

95. $25a^4 - 41a^2b^2 + 16b^4$.

96. $4x^4 - 37x^2y^2 + 9y^4$.

97. $49m^4 - 11m^2n^2 + 25n^4$.

98. $36x^4 - 16x^2y^2 + 49y^4$.

99. $81a^4 + 20a^2b^2 + 4b^4$.

100. $25a^4 - 49a^2y^2 + 16y^4$.

101. $4x^4 - 28x^2a^2 + 9a^4$.

102. $a^4 + 4$.

103. $x^4 + 64$.

104. $a^4 + 4x^4$.

105. $4a^4 + 16x^4$.

106. $x^8 + 324$.

107. $a^4b^4 + 4c^4$.

108. $x^8 + 1024$.

109. $x^{12} + 2500$.

110. $a^8x^4 + 4b^4$.

93. A **Theorem** is a truth to be demonstrated. A *Demonstration* is a course of reasoning employed in proving a theorem.

94. When the binomial is the sum or difference of any powers of two quantities.

This case relates to binomials of the form of $a^3 - b^3$, $a^3 + b^3$, $a^4 - b^4$, $a^6 + b^6$, $a^9 + b^6$, $a^3 + b^{12}$, etc.

THEOREM I.

The difference of any powers of two quantities is divisible by the **difference** *of the* **same** *roots of those powers.*

ILLUSTRATION.

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2.$$
$$(a^9 - b^6) \div (a^3 - b^2) = a^6 + a^3b^2 + b^4.$$
$$(a^4 - b^{12}) \div (a - b^3) = a^3 + a^2b^3 + ab^6 + b^9.$$

DEMONSTRATION.—Let $a^n - b^n$ be the difference of any powers of two quantities, and $a - b$ the difference of the same roots of those powers. Then is $a^n - b^n$ divisible by $a - b$.

Dividing until several remainders are obtained, we have

$$\begin{array}{rl|l}
 & a^n - \qquad b^n & a - b \\
 & \underline{a^n - a^{n-1}b} & \underline{a^{n-1} + a^{n-2}b} \\
\text{1st Rem.} = & a^{n-1}b - b^n & \\
 & \underline{a^{n-1}b - a^{n-2}b^2} & \\
\text{2d Rem.} = & a^{n-2}b^2 - b^n & \\
n\text{th Rem.} = & a^{n-n}b^n - b^n & \\
= & a^0b^n - b^n & \\
= & b^n - b^n = 0. &
\end{array}$$

It is seen that the first term of the first remainder is $a^{n-1}b$; of the second remainder, $a^{n-2}b^2$; of the third remainder, $a^{n-3}b^3$; of the fourth remainder, $a^{n-4}b^4$. Hence the nth remainder will be $a^{n-n}b^n$; but $a^{n-n} = a^0 = 1$ (ART. 80, Prin. 4). Therefore, the first term of the nth remainder reduces to b^n.

Since the second term of the nth remainder is $-b^n$, the nth remainder is $b^n - b^n = 0$; that is, there is no remainder, and the division terminates. Therefore, $a^n - b^n$ is divisible by $a - b$.

EXAMPLES.

Factor—

1. $a^3 - b^3$.
2. $a^4 - b^4$.
3. $a^5 - b^5$.
4. $a^7 - b^7$.
5. $a^{11} - b^{11}$.
6. $x^6 - y^9$.
7. $x^9 - y^3$.
8. $x^5 - y^{10}$.
9. $x^{10} - y^{15}$.
10. $x^9 - y^{12}$.
11. $m^3 - 125$.
12. $n^5 - 32$.
13. $128 - a^7$.
14. $a^{11} - 1$.
15. $a^{3n} - b^{3m}$.

THEOREM II.

The difference of any powers of two quantities is divisible by the **sum** *of the same* **even** *roots of those powers.*

ILLUSTRATION.

$$(a^2 - b^2) \div (a + b) = a - b.$$
$$(a^8 - b^4) \div (a^2 + b) = a^6 - a^4b + a^2b^2 - b^3.$$
$$(a^4 - b^{12}) \div (a + b^3) = a^3 - a^2b^3 + ab^6 - b^9.$$

DEMONSTRATION.—Let $a^n - b^n$ be the difference of any powers of two quantities, and $a + b$ the sum of the same even roots of those powers. Then is $a^n - b^n$ divisible by $a + b$.

Dividing until four remainders are obtained, we have

$$\begin{array}{l}
a^n - \qquad b^n \;\Big|\; \underline{a + b} \\
\underline{a^n + a^{n-1}b \;\Big|\; a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3} \\
\text{1st Rem.} = -a^{n-1}b - b^n \\
\qquad \underline{-a^{n-1}b - a^{n-2}b^2} \\
\qquad \text{2d Rem.} = +a^{n-2}b^2 - b^n \\
\qquad\qquad \underline{+a^{n-2}b^2 + a^{n-3}b^3} \\
\qquad\qquad \text{3d Rem.} = -a^{n-3}b^3 - b^n \\
\qquad\qquad\qquad \underline{-a^{n-3}b^3 - a^{n-4}b^4} \\
\qquad\qquad\qquad \text{4th Rem.} = +a^{n-4}b^4 - b^n \\
\text{If } n \text{ is } \textit{even}, \text{ the } n\text{th Rem.} = +a^{n-n}b^n - b^n \\
\qquad\qquad\qquad\qquad = +a^0b^n - b^n \\
\qquad\qquad\qquad\qquad = b^n - b^n = 0.
\end{array}$$

It is seen that the first term of the odd remainders is $-$, and of the even remainders is $+$. Hence if n is even, the first term of the nth remainder is $+a^{n-n}b^n$; but $a^{n-n} = a^0 = 1$ (ART. 80, Prin. 4). Therefore, the first term of the nth remainder reduces to b^n.

Since the second term of the nth remainder is $-b^n$, the nth remainder is $b^n - b^n = 0$; that is, there is no remainder, and the division terminates. Therefore, $a^n - b^n$ is divisible by $a + b$ if n is even.

EXAMPLES.

Factor—

1. $a^2 - b^4$.
2. $a^4 - b^4$.
3. $x^6 - b^6$.
4. $a^8 - b^8$.
5. $m^{10} - n^{10}$.
6. $16 - a^4$.
7. $a^4 - 1$.
8. $x^8 - 1$.
9. $a^4 - b^{16}$.
10. $a^4b^8 - 1$.
11. $81 - y^4$.
12. $a^6x^6 - y^{12}$.
13. $x^{10} - y^{20}$.
14. $a^{4n} - b^{4n}$.
15. $x^{8n} - y^{4n}$.

THEOREM III.

The sum of any powers of two quantities is divisible by the **sum** *of the same* **odd** *roots of those powers.*

ILLUSTRATION.

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2.$$
$$(a^5 + b^{10}) \div (a + b^2) = a^4 - a^3b^2 + a^2b^4 - ab^6 + b^8.$$
$$(a^6 + b^6) \div (a^2 + b^2) = a^4 - a^2b^2 + b^4.$$

DEMONSTRATION.—Let $a^n + b^n$ be the sum of any two powers of two quantities, and $a + b$ the sum of the same odd roots of those powers. Then is $a^n + b^n$ divisible by $a + b$.

Dividing until four remainders are obtained, we have

$$
\begin{array}{r}
a^n + \qquad b^n \;\big|\; \underline{a + b \qquad\qquad\qquad\qquad\qquad} \\
\underline{a^n + a^{n-1}b \;\big|\; a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3} \\
\text{1st Rem.} = -a^{n-1}b + b^n \qquad\qquad\qquad\qquad\qquad\qquad \\
\underline{-a^{n-1}b - a^{n-2}b^2} \qquad\qquad\qquad\qquad\qquad \\
\text{2d Rem.} = a^{n-2}b^2 + b^n \qquad\qquad\qquad\qquad \\
\underline{+a^{n-2}b^2 + a^{n-3}b^3} \qquad\qquad\qquad \\
\text{3d Rem.} = -a^{n-3}b^3 + b^n \qquad\qquad\qquad \\
\underline{-a^{n-3}b^3 - a^{n-4}b^4} \\
\text{4th Rem.} = a^{n-4}b^4 + b^n \\
\text{If } n \text{ is } \textit{odd}, \text{ the } n\text{th Rem.} = -a^{n-n}b^n + b^n \\
= -a^0b^n + b^n \\
= -b^n + b^n = 0.
\end{array}
$$

It is seen that the first term of the odd remainders is $-$, and of the even remainders is $+$. Hence if n is odd the first term of the nth remainder is $-a^{n-n}b^n$; but $a^{n-n} = a^0 = 1$ (ART. 80, Prin. 4). Therefore, the first term of the nth remainder reduces to $-b^n$.

Since the second term of the nth remainder is $+ b^n$, the nth remainder is $- b^n + b^n = 0$; that is, there is no remainder, and the division terminates. Therefore, $a^n + b^n$ is divisible by $a + b$ if n is odd.

EXAMPLES.

Factor—

1. $a^5 + x^5$.
2. $a^7 + y^7$.
3. $a^9 + b^9$
4. $a^6 + n^6$.
5. $n^{10} + m^{10}$.
6. $x^{12} + y^{12}$.
7. $a^{14} + y^{14}$.
8. $x^{15} + y^{10}$.
9. $a^{14} + x^7$.
10. $m^6 + n^9$.
11. $a^{15} + n^6$.
12. $n^3 + 27$.
13. $a^5 + b^{10}c^5$.
14. $1 + n^7$.
15. $a^{3n} + b^{12n}$.

9. $(x - y + z)(x - y - z)$.

10. $(a^2 - ab + b^2)(a^2 + ab + b^2)$.

11. $(m^2 + mn + n^2)(m^2 - mn + n^2)$.

12. $(a^2 + a + 1)(a^2 - a - 1)$.

13. $(x^2 - xy + y^2)(x^2 + xy + y^2)$.

14. $(1 - n + n^2)(1 + n + n^2)$.

15. $(3a^2 + 3a - 1)(3a^2 - 3a + 1)$.

16. $(m - n)(m + n)(m^2 + n^2)(m^4 + n^4)$.

17. $(1 - a)(1 + a)(1 + a^2)(1 + a^4)(1 + a^8)$.

18. $(a - x)(a + x)(a^2 + x^2)(a^4 + x^4)(a^8 + x^8)$.

19. $(a + b + m + n)(a + b - m - n)$.

20. $(a + m - b - n)(a - m - b - n)$.

21. $(a^n - b^n)(a^n + b^n)\ a^{2n} + b^{2n})(a^{4n} + b^{4n})$.

22. $(a^n - y^n + b^n + x^n)(a^n - y^n - b^n - x^n)$.

FACTORING.

88. Factoring is the process of resolving a composite quantity into its factors.

The *Factors* of a composite quantity are the quantities which when multiplied together will produce it.

89. A **Prime Quantity** is one that cannot be produced by the multiplication of other quantities; as, 17, a, $a + b$, or $a^2 + b^2$.

In speaking of prime quantities in algebra we must exclude factoring by means of *fractional exponents* and *imaginary quantities;** otherwise many quantities regarded as prime could be factored.

For example, $a - b = (a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.

Or, $\qquad = (a^{\frac{1}{3}} - b^{\frac{1}{3}})(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})$, etc.

$$a^2 + b^2 = (a + b\sqrt{-1})(a - b\sqrt{-1}).$$

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* An imaginary quantity is the indicated even root of a negative quantity; as, $\sqrt{-a}$, or $\sqrt[2n]{-a}$.

90. By Art. 8, one of the two equal factors of a quantity is its *square root.*

Thus, $2a$ is the square root of $4a^2$.

I. TO FACTOR MONOMIALS.

91. The prime factors of monomials may be obtained by inspection, and therefore need no further consideration.

Thus, the prime factors of $6a^2b^3 = 2 \times 3aabbb$.

II. TO FACTOR BINOMIALS.

92. When the binomial is the difference of two squares.

1. Factor $a^2 - b^2$ and $4x^2 - 9y^2$.

Referring to Art. 86, Prin. III., we see that such binomials are the product of the sum and difference of two quantities.

Hence $a^2 - b^2 = (a + b)(a - b)$.

$4x^2 - 9y^2 = (2x + 3y)(2x - 3y)$.

Therefore, to obtain the factors use the following

RULE.

Extract the square root of each term of the binomial, and make their sum one factor and their difference the other factor.

Note.—This is one of the most general and useful cases of factoring, and should always be used first when applicable.

EXAMPLES.

Factor—

2. $m^2 - n^2$.
3. $c^2 - d^2$.
4. $x^2 - y^2$.
5. $p^2 - q^2$.
6. $x^2 - 1$.
7. $4 - y^2$.
8. $x^2 - 4y^2$.
9. $4a^2 - 9b^2$.
10. $16m^2 - 25n^2$.
11. $36x^2 - 49a^2$.
12. $36l^2 - 64m^2$.
13. $x^2y^2 - a^2b^2$.
14. $m^4 - n^4$.
15. $a^8 - b^8$.
16. $a^4 - x^8$.
17. $m^{16} - n^{16}$.
18. $x^{16} - y^8$.
19. $a^{2n} - b^{2m}$.
20. $9a^{2n} - 16b^{2n}$.
21. $a^{14} - b^4$.
22. $x^{8n} - y^{8m}$.

23. $\frac{1}{4}x^2 - \frac{1}{9}y^2$.

24. $225x^2y^2 - 9$.

25. $289x^{2n} - 121y^{2n}$.

26. $81m^2 - 100n^6$.

27. $a^2b^6 - c^2d^8$.

28. $\frac{1}{16}x^4 - 1$.

29. $144 - a^2b^{12}$.

30. $a^{32} - b^{32}$.

31. $a^{16n} - b^{16n}$.

The same principle may be applied to the following binomial forms:

32. Factor $a^2 - (b + c)^2$.

$$\text{1st factor} = [a + (b + c)] = (a + b + c).$$
$$\text{2d factor} = [a - (b + c)] = (a - b - c).$$

33. Factor $(a + b)^2 - (c + d)^2$.

$$\text{1st factor} = [(a + b) + (c + d)] = (a + b + c + d).$$
$$\text{2d factor} = [(a + b) - (c + d)] = (a + b - c - d).$$

34. $(a + b)^2 - c^2$.

35. $(m + n)^2 - p^2$.

36. $(x + y)^2 - z^2$.

37. $m^2 - (x - y)^2$.

38. $(a - b)^2 - c^2$.

39. $a^2 - (c + x)^2$.

40. $4 - (a - b)^2$.

41. $(x - y)^2 - 9$.

42. $m^4 - (m^2 - 1)^2$.

43. $x^2 - (x + y)^2$.

44. $(3x - 5)^2 - 4x^2$.

45. $m^6 - (m^3 + 1)^2$.

46. $(x + y)^2 - (z + v)^2$.

47. $(m + n)^2 - (x - y)^2$.

48. $(a - x)^2 - (y + b)^2$.

49. $(m - n)^2 - (p - q)^2$.

50. $(x + y)^2 - (x - y)^2$.

51. $(a - b)^2 - (a + b)^2$.

52. $(2a - c)^2 - (3b - x)^2$.

53. $(2x - 4)^2 - (x - 4)^2$.

54. $(5x + 2y)^2 - (5 - 3y)^2$.

55. $(x - 1)^2 - (y + 1)^2$.

56. $(ax + by)^2 - (ax - by)^2$.

57. $(m^2 + n^2)^2 - (a^2 - b^2)^2$.

Polynomials can sometimes be arranged in the form of the difference of two squares, and factored like the preceding examples.

58. Factor $x^2 + 2xy + y^2 - z^2$.

This may be arranged and factored as follows:

$$x^2 + 2xy + y^2 - z^2 = (x + y)^2 - z^2 = (x + y + z)(x + y - z).$$

59. Factor $a^2 + b^2 - c^2 - d^2 - 2cd - 2ab$.

This may be factored thus:

$$\begin{aligned}
&a^2 + b^2 - c^2 - d^2 - 2cd - 2ab \\
&= a^2 - 2ab + b^2 - c^2 - 2cd - d^2 \\
&= (a^2 - 2ab + b^2) - (c^2 + 2cd + d^2) \\
&= (a - b)^2 - (c + d)^2 \\
&= [(a - b) + (c + d)]\,[(a - b) - (c + d)] \\
&= (a - b + c + d)\,(a - b - c - d).
\end{aligned}$$

60. $a^2 + 2ab + b^2 - c^2$.

61. $x^2 - 2xy + y^2 - z^2$.

62. $x^2 - a^2 - 2ab - b^2$.

63. $y^2 - m^2 + 2mn - n^2$.

64. $1 - m^2 - n^2 - 2mn$.

65. $2ab - a^2 - b^2 + 1$.

66. $a^2 - 2xy - x^2 - y^2$.

67. $a^2 - y^2 + b^2 - 2ab$.

68. $x^2 - 2yz + y^2 - z^2$.

69. $m^2 - 2mn + n^2 - p^2$.

70. $m^2 - n^2 + 2pn - p^2$.

71. $m^2 - n^2 - 2pn - p^2$.

72. $m^2 + n^2 - p^2 + 2mn$.

73. $x^2 - 1 + y^2 + 2xy$.

74. $a^2 + 2ab + b^2 - x^2 + 2xy - y^2$.

75. $m^2 - 2mn + n^2 - p^2 - 2pq - q^2$.

76. $a^2 - y^2 - b^2 + x^2 - 2ax + 2by$.

77. $a^2 - 9y^2 - 6xy + 4b^2 - 4ab - x^2$.

78. $a^2 - 9y^2 + 9x^2 - 4z^2 + 12yz + 6ax$.

79. $m^2 - 2xy + n^2 - x^2 + 2mn - y^2$.

Sometimes a quantity may be both *added to*, and *subtracted from*, a polynomial, and the terms then arranged in the form of the difference of two squares and factored like the preceding examples.

80. Factor $x^4 + x^2y^2 + y^4$.

Since $x^4 + x^2y^2 + y^4$ lacks x^2y^2 of being a perfect square, if x^2y^2 be added *to* the expression and then subtracted *from* it, the value will not be changed (ART. 54, Axiom 5).

Adding and subtracting x^2y^2, the expression becomes

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2 \\ &= [(x^2 + y^2) + xy]\,[(x^2 + y^2) - xy] \\ &= (x^2 + xy + y^2)\,(x^2 - xy + y^2). \end{aligned}$$

81. Factor $a^4 + a^2 + 1$.

Adding and subtracting a^2,

$$\begin{aligned} a^4 + a^2 + 1 &= a^4 + 2a^2 + 1 - a^2 \\ &= (a^2 + 1)^2 - a^2 \\ &= (a^2 + a + 1)\,(a^2 - a + 1). \end{aligned}$$

82. $m^4 + m^2n^2 + n^4$.
83. $1 + y^2 + y^4$.
84. $b^4 + b^2c^2 + c^4$.
85. $a^4 + a^2x^2 + x^4$.
86. $x^4 - 3x^2y^2 + y^4$.
87. $a^4 - 6a^2n^2 + n^4$.
88. $a^4 - 7a^2y^2 + y^4$.
89. $m^4 - 27m^2n^2 + n^4$.
90. $l^4 - 38l^2m^2 + m^4$.
91. $4x^4 - 16x^2y^2 + 9y^4$.
92. $9a^4 + 14a^2b^2 + 25b^4$.
93. $25a^4 + a^2x^2 + x^4$.
94. $25x^4 - 79x^2y^2 + 49y^4$.
95. $25a^4 - 41a^2b^2 + 16b^4$.
96. $4x^4 - 37x^2y^2 + 9y^4$.
97. $49m^4 - 11m^2n^2 + 25n^4$.
98. $36x^4 - 16x^2y^2 + 49y^4$.
99. $81a^4 + 20a^2b^2 + 4b^4$.
100. $25a^4 - 49a^2y^2 + 16y^4$.
101. $4x^4 - 28x^2a^2 + 9a^4$.

102. $a^4 + 4$.
103. $x^4 + 64$.
104. $a^4 + 4x^4$.
105. $4a^4 + 16x^4$.
106. $x^8 + 324$.
107. $a^4b^4 + 4c^4$.
108. $x^8 + 1024$.
109. $x^{12} + 2500$.
110. $a^8x^4 + 4b^4$.

93. A **Theorem** is a truth to be demonstrated. A *Demonstration* is a course of reasoning employed in proving a theorem.

94. **When the binomial is the sum or difference of any powers of two quantities.**

This case relates to binomials of the form of $a^3 - b^3$, $a^3 + b^3$, [illegible]$^4 - b^4$, $a^6 + b^6$, $a^9 + b^6$, $a^3 + b^{12}$, etc.

THEOREM I.

The difference of any powers of two quantities is divisible by the **difference** *of the* **same** *roots of those powers.*

ILLUSTRATION.

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2.$$
$$(a^9 - b^6) \div (a^3 - b^2) = a^6 + a^3b^2 + b^4.$$
$$(a^4 - b^{12}) \div (a - b^3) = a^3 + a^2b^3 + ab^6 + b^9.$$

DEMONSTRATION.—Let $a^n - b^n$ be the difference of any powers of two quantities, and $a - b$ the difference of the same roots of those powers. Then is $a^n - b^n$ divisible by $a - b$.

Dividing until several remainders are obtained, we have

$$\begin{array}{rl|l}
 & a^n - \qquad b^n & a - b \\ \cline{3-3}
 & \underline{a^n - a^{n-1}b} & \underline{a^{n-1} + a^{n-2}b} \\
\text{1st Rem.} = & a^{n-1}b - b^n & \\
 & \underline{a^{n-1}b - a^{n-2}b^2} & \\
\text{2d Rem.} = & a^{n-2}b^2 - b^n & \\
n\text{th Rem.} = & a^{n-n}b^n - b^n & \\
= & a^0b^n - b^n & \\
= & b^n - b^n = 0. &
\end{array}$$

It is seen that the first term of the first remainder is $a^{n-1}b$; of the second remainder, $a^{n-2}b^2$; of the third remainder, $a^{n-3}b^3$; of the fourth remainder, $a^{n-4}b^4$. Hence the nth remainder will be $a^{n-n}b^n$; but $a^{n-n} = a^0 = 1$ (ART. 80, Prin. 4). Therefore, the first term of the nth remainder reduces to b^n.

Since the second term of the nth remainder is $-b^n$, the nth remainder is $b^n - b^n = 0$; that is, there is no remainder, and the division terminates. Therefore, $a^n - b^n$ is divisible by $a - b$.

EXAMPLES.

Factor—

1. $a^3 - b^3$.
2. $a^4 - b^4$.
3. $a^5 - b^5$.
4. $a^7 - b^7$.
5. $a^{11} - b^{11}$.
6. $x^6 - y^9$.
7. $x^9 - y^3$.
8. $x^5 - y^{10}$.
9. $x^{10} - y^{15}$.
10. $x^9 - y^{12}$.
11. $m^3 - 125$.
12. $n^5 - 32$.
13. $128 - a^7$.
14. $a^{11} - 1$.
15. $a^{3n} - b^{3m}$.

THEOREM II.

The difference of any powers of two quantities is divisible by the **sum** *of the same* **even** *roots of those powers.*

ILLUSTRATION.

$$(a^2 - b^2) \div (a + b) = a - b.$$
$$(a^8 - b^4) \div (a^2 + b) = a^6 - a^4b + a^2b^2 - b^3.$$
$$(a^4 - b^{12}) \div (a + b^3) = a^3 - a^2b^3 + ab^6 - b^9.$$

DEMONSTRATION.—Let $a^n - b^n$ be the difference of any powers of two quantities, and $a + b$ the sum of the same even roots of those powers. Then is $a^n - b^n$ divisible by $a + b$.

Dividing until four remainders are obtained, we have

$$\begin{array}{rl}
a^n - \qquad b^n & \big| \; a + b \\
\underline{a^n + a^{n-1}b} & \big| \; \overline{a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3} \\
\text{1st Rem.} = & - a^{n-1}b - b^n \\
 & \underline{- a^{n-1}b - a^{n-2}b^2} \\
\text{2d Rem.} = & + a^{n-2}b^2 - b^n \\
 & \underline{+ a^{n-2}b^2 + a^{n-3}b^3} \\
\text{3d Rem.} = & - a^{n-3}b^3 - b^n \\
 & \underline{- a^{n-3}b^3 - a^{n-4}b^4} \\
\text{4th Rem.} = & + a^{n-4}b^4 - b^n \\
\text{If } n \text{ is } \textit{even}, \text{ the } n\text{th Rem.} = & + a^{n-n}b^n - b^n \\
= & + a^0b^n - b^n \\
= & b^n - b^n = 0.
\end{array}$$

It is seen that the first term of the odd remainders is $-$, and of the even remainders is $+$. Hence if n is even, the first term of the nth remainder is $+ a^{n-n}b^n$; but $a^{n-n} = a^0 = 1$ (ART. 80, Prin. 4). Therefore, the first term of the nth remainder reduces to b^n.

Since the second term of the nth remainder is $- b^n$, the nth remainder is $b^n - b^n = 0$; that is, there is no remainder, and the division terminates. Therefore, $a^n - b^n$ is divisible by $a + b$ if n is even.

EXAMPLES.

Factor—

1. $a^2 - b^4$.
2. $a^4 - b^4$.
3. $x^6 - b^6$.
4. $a^8 - b^8$.
5. $m^{10} - n^{10}$.
6. $16 - a^4$.
7. $a^4 - 1$.
8. $x^8 - 1$.
9. $a^4 - b^{16}$.
10. $a^4b^8 - 1$.
11. $81 - y^4$.
12. $a^6x^6 - y^{12}$.
13. $x^{10} - y^{20}$.
14. $a^{4n} - b^{4n}$.
15. $x^{8n} - y^{4n}$.

THEOREM III.

The sum of any powers of two quantities is divisible by the **sum** *of the same* **odd** *roots of those powers.*

ILLUSTRATION.

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2.$$

$$(a^5 + b^{10}) \div (a + b^2) = a^4 - a^3b^2 + a^2b^4 - ab^6 + b^8.$$

$$(a^6 + b^6) \div (a^2 + b^2) = a^4 - a^2b^2 + b^4.$$

DEMONSTRATION.—Let $a^n + b^n$ be the sum of any two powers of two quantities, and $a + b$ the sum of the same odd roots of those powers. Then is $a^n + b^n$ divisible by $a + b$.

Dividing until four remainders are obtained, we have

$$\begin{array}{r|l}
a^n + \qquad b^n & a + b \\
\underline{a^n + a^{n-1}b} & \overline{a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3} \\
\text{1st Rem.} = -a^{n-1}b + b^n & \\
\underline{-a^{n-1}b - a^{n-2}b^2} & \\
\text{2d Rem.} = a^{n-2}b^2 + b^n & \\
\underline{+a^{n-2}b^2 + a^{n-3}b^3} & \\
\text{3d Rem.} = -a^{n-3}b^3 + b^n & \\
\underline{-a^{n-3}b^3 - a^{n-4}b^4} & \\
\text{4th Rem.} = a^{n-4}b^4 + b^n & \\
\text{If } n \text{ is } \textit{odd}\text{, the } n\text{th Rem.} = -a^{n-n}b^n + b^n & \\
= -a^0b^n + b^n & \\
= -b^n + b^n = 0. &
\end{array}$$

It is seen that the first term of the odd remainders is $-$, and of the even remainders is $+$. Hence if n is odd the first term of the nth remainder is $-a^{n-n}b^n$; but $a^{n-n} = a^0 = 1$ (ART. 80, Prin. 4). Therefore, the first term of the nth remainder reduces to $-b^n$.

Since the second term of the nth remainder is $+ b^n$, the nth remainder is $- b^n + b^n = 0$; that is, there is no remainder, and the division terminates. Therefore, $a^n + b^n$ is divisible by $a + b$ if n is odd.

EXAMPLES.

Factor—

1. $a^5 + x^5$.
2. $a^7 + y^7$.
3. $a^9 + b^9$
4. $a^6 + n^6$.
5. $n^{10} + m^{10}$.
6. $x^{12} + y^{12}$.
7. $a^{14} + y^{14}$.
8. $x^{15} + y^{10}$.
9. $a^{14} + x^7$.
10. $m^6 + n^9$.
11. $a^{15} + n^6$.
12. $n^3 + 27$.
13. $a^5 + b^{10}c^5$.
14. $1 + n^7$.
15. $a^{3n} + b^{12n}$.

MISCELLANEOUS EXAMPLES.

16. $a^3 - 343$.
17. $32x^5 - y^5$.
18. $a^5 + x^{10}y^{15}$.
19. $a^6 - b^6$.
20. $a^{12} - b^{12}$.
21. $x^{16} - y^{16}$.
22. $a^{10}y^5 - x^5b^{10}$.
23. $a^5b^{10} - x^{10}y^5$.
24. $a^6 - 1$.
25. $x^{14} - y^7$.
26. $x^4 - y^4$.
27. $a^5b^5 - x^{15}y^{15}$.
28. $x^7 + y^{21}$.
29. $a^4 - b^{16}$.
30. $a^9 - b^3$.

III. TO FACTOR TRINOMIALS.

95. A **Quadratic Trinomial** is one whose form is $x^2 \pm ax \pm b$, in which b is the product of two quantities and a their algebraic sum.

96. When the trinomial is quadratic.

A quadratic trinomial will assume one of the six following forms:

FIRST FORM.—Factor $a^2 + 2ab + b^2$.

By ART. 86, Prin. I., this is evidently the square of the sum of a and b. Hence,

$$a^2 + 2ab + b^2 = (a + b)(a + b) = (a + b)^2.$$

SECOND FORM.—Factor $a^2 - 2ab + b^2$.

By Prin. II., ART. 86, this is evidently the square of the difference of a and b. Hence,

$$a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2.$$

THIRD FORM.—Factor $a^2 + 7a + 12$.

By Prin. IV., ART. 86, the first term of each binomial is a, the product of the other two terms is 12, and their sum is 7. The factors of 12 are

$$1 \times 12,$$
$$2 \times 6,$$
$$3 \times 4.$$

Since the sum of the two terms is 7, they must be 3 and 4. Hence,

$$a^2 + 7a + 12 = (a + 3)(a + 4).$$

FOURTH FORM.—Factor $a^2 - 7a + 12$.

This form is factored like the Third Form, with the second term in each factor *negative*. Hence,

$$a^2 - 7a + 12 = (a - 3)(a - 4).$$

FIFTH FORM.—Factor $a^2 + a - 12$.

This form is factored like the Third Form, with the second term in one factor *positive* and the other *negative*, the positive term being the greater. Hence,

$$a^2 + a - 12 = (a - 3)(a + 4).$$

SIXTH FORM.—Factor $a^2 - a - 12$.

This form is factored like the Third Form, with the second term in one factor *positive* and the other *negative*, the negative term being the greater. Hence,

$$a^2 - a - 12 = (a + 3)(a - 4).$$

EXAMPLES.

Factor—

1. $m^2 + 2mn + n^2$.
2. $a^2 - 8a + 16$.
3. $x^2 + 3x + 2$.
4. $a^2 - 4a + 3$.
5. $a^2 + a - 2$.
6. $n^4 - n^2 - 2$.
7. $x^2 + 12x + 36$.
8. $m^8 - 34m^4 + 289$.
9. $y^2 + 6y + 5$.
10. $n^2 - 12n + 11$.
11. $x^2 + x - 6$.
12. $n^2 - n - 20$.
13. $a^2 + 8a + 16$.
14. $s^2 - 42s + 441$.
15. $m^2 + 5m + 6$.
16. $x^2 - 16x + 15$.
17. $m^2 + m - 12$.
18. $x^2 - x - 42$.
19. $y^2 + 10y + 25$.
20. $1 - 44a^2 + 484a^4$.
21. $x^2 + 13x + 22$.
22. $z^2 - 12z + 20$.
23. $n^2 + n - 20$.
24. $z^4 - 13z^2 - 140$.
25. $a^2 + 4a + 4$.
26. $a^2 - 16ab + 64b^2$.
27. $n^2 + 43n + 82$.
28. $x^8 - 11x^4 + 28$.

23. $\frac{1}{4}x^2 - \frac{1}{9}y^2$.

24. $225x^2y^2 - 9$.

25. $289x^{2n} - 121y^{2n}$.

26. $81m^2 - 100n^6$.

27. $a^2b^6 - c^2d^8$.

28. $\frac{1}{16}x^4 - 1$.

29. $144 - a^2b^{12}$.

30. $a^{32} - b^{32}$.

31. $a^{16n} - b^{16n}$.

The same principle may be applied to the following binomial forms:

32. Factor $a^2 - (b + c)^2$.

$$\text{1st factor} = [a + (b + c)] = (a + b + c).$$
$$\text{2d factor} = [a - (b + c)] = (a - b - c).$$

33. Factor $(a + b)^2 - (c + d)^2$.

$$\text{1st factor} = [(a + b) + (c + d)] = (a + b + c + d).$$
$$\text{2d factor} = [(a + b) - (c + d)] = (a + b - c - d).$$

34. $(a + b)^2 - c^2$.

35. $(m + n)^2 - p^2$.

36. $(x + y)^2 - z^2$.

37. $m^2 - (x - y)^2$.

38. $(a - b)^2 - c^2$.

39. $a^2 - (c + x)^2$.

40. $4 - (a - b)^2$.

41. $(x - y)^2 - 9$.

42. $m^4 - (m^2 - 1)^2$.

43. $x^2 - (x + y)^2$.

44. $(3x - 5)^2 - 4x^2$.

45. $m^6 - (m^3 + 1)^2$.

46. $(x + y)^2 - (z + v)^2$.

47. $(m + n)^2 - (x - y)^2$.

48. $(a - x)^2 - (y + b)^2$.

49. $(m - n)^2 - (p - q)^2$.

50. $(x + y)^2 - (x - y)^2$.

51. $(a - b)^2 - (a + b)^2$.

52. $(2a - c)^2 - (3b - x)^2$.

53. $(2x - 4)^2 - (x - 4)^2$.

54. $(5x + 2y)^2 - (5 - 3y)^2$.

55. $(x - 1)^2 - (y + 1)^2$.

56. $(ax + by)^2 - (ax - by)^2$.

57. $(m^2 + n^2)^2 - (a^2 - b^2)^2$.

Polynomials can sometimes be arranged in the form of the difference of two squares, and factored like the preceding examples.

58. Factor $x^2 + 2xy + y^2 - z^2$.

This may be arranged and factored as follows:

$$x^2 + 2xy + y^2 - z^2 = (x + y)^2 - z^2 = (x + y + z)(x + y - z).$$

59. Factor $a^2 + b^2 - c^2 - d^2 - 2cd - 2ab$.

This may be factored thus:

$$\begin{aligned} &a^2 + b^2 - c^2 - d^2 - 2cd - 2ab \\ &= a^2 - 2ab + b^2 - c^2 - 2cd - d^2 \\ &= (a^2 - 2ab + b^2) - (c^2 + 2cd + d^2) \\ &= (a-b)^2 - (c+d)^2 \\ &= [(a-b) + (c+d)]\,[(a-b) - (c+d)] \\ &= (a - b + c + d)\,(a - b - c - d). \end{aligned}$$

60. $a^2 + 2ab + b^2 - c^2$.

61. $x^2 - 2xy + y^2 - z^2$.

62. $x^2 - a^2 - 2ab - b^2$.

63. $y^2 - m^2 + 2mn - n^2$.

64. $1 - m^2 - n^2 - 2mn$.

65. $2ab - a^2 - b^2 + 1$.

66. $a^2 - 2xy - x^2 - y^2$.

67. $a^2 - y^2 + b^2 - 2ab$.

68. $x^2 - 2yz + y^2 - z^2$.

69. $m^2 - 2mn + n^2 - p^2$.

70. $m^2 - n^2 + 2pn - p^2$.

71. $m^2 - n^2 - 2pn - p^2$.

72. $m^2 + n^2 - p^2 + 2mn$.

73. $x^2 - 1 + y^2 + 2xy$.

74. $a^2 + 2ab + b^2 - x^2 + 2xy - y^2$.

75. $m^2 - 2mn + n^2 - p^2 - 2pq - q^2$.

76. $a^2 - y^2 - b^2 + x^2 - 2ax + 2by$.

77. $a^2 - 9y^2 - 6xy + 4b^2 - 4ab - x^2$.

78. $a^2 - 9y^2 + 9x^2 - 4z^2 + 12yz + 6ax$.

79. $m^2 - 2xy + n^2 - x^2 + 2mn - y^2$.

Sometimes a quantity may be both *added to*, and *subtracted from*, a polynomial, and the terms then arranged in the form of the difference of two squares and factored like the preceding examples.

80. Factor $x^4 + x^2y^2 + y^4$.

Since $x^4 + x^2y^2 + y^4$ lacks x^2y^2 of being a perfect square, if x^2y^2 be added *to* the expression and then subtracted *from* it, the value will not be changed (ART. 54, Axiom 5).

Adding and subtracting x^2y^2, the expression becomes

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2 \\ &= [(x^2 + y^2) + xy]\,[(x^2 + y^2) - xy] \\ &= (x^2 + xy + y^2)\,(x^2 - xy + y^2). \end{aligned}$$

81. Factor $a^4 + a^2 + 1$.

Adding and subtracting a^2,

$$\begin{aligned} a^4 + a^2 + 1 &= a^4 + 2a^2 + 1 - a^2 \\ &= (a^2 + 1)^2 - a^2 \\ &= (a^2 + a + 1)\,(a^2 - a + 1). \end{aligned}$$

82. $m^4 + m^2n^2 + n^4$.

83. $1 + y^2 + y^4$.

84. $b^4 + b^2c^2 + c^4$.

85. $a^4 + a^2x^2 + x^4$.

86. $x^4 - 3x^2y^2 + y^4$.

87. $a^4 - 6a^2n^2 + n^4$.

88. $a^4 - 7a^2y^2 + y^4$.

89. $m^4 - 27m^2n^2 + n^4$.

90. $l^4 - 38l^2m^2 + m^4$.

91. $4x^4 - 16x^2y^2 + 9y^4$.

92. $9a^4 + 14a^2b^2 + 25b^4$.

93. $25a^4 + a^2x^2 + x^4$.

94. $25x^4 - 79x^2y^2 + 49y^4$.

95. $25a^4 - 41a^2b^2 + 16b^4$.

96. $4x^4 - 37x^2y^2 + 9y^4$.

97. $49m^4 - 11m^2n^2 + 25n^4$.

98. $36x^4 - 16x^2y^2 + 49y^4$.

99. $81a^4 + 20a^2b^2 + 4b^4$.

100. $25a^4 - 49a^2y^2 + 16y^4$.

101. $4x^4 - 28x^2a^2 + 9a^4$.

102. $a^4 + 4$.

103. $x^4 + 64$.

104. $a^4 + 4x^4$.

105. $4a^4 + 16x^4$.

106. $x^8 + 324$.

107. $a^4b^4 + 4c^4$.

108. $x^8 + 1024$.

109. $x^{12} + 2500$.

110. $a^8x^4 + 4b^4$.

93. A **Theorem** is a truth to be demonstrated. A *Demonstration* is a course of reasoning employed in proving a theorem.

94. When the binomial is the sum or difference of any powers of two quantities.

This case relates to binomials of the form of $a^3 - b^3$, $a^3 + b^3$, $a^4 - b^4$, $a^6 + b^6$, $a^9 + b^6$, $a^3 + b^{12}$, etc.

THEOREM I.

The difference of any powers of two quantities is divisible by the **difference** *of the* **same** *roots of those powers.*

ILLUSTRATION.

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2.$$
$$(a^9 - b^6) \div (a^3 - b^2) = a^6 + a^3b^2 + b^4.$$
$$(a^4 - b^{12}) \div (a - b^3) = a^3 + a^2b^3 + ab^6 + b^9.$$

DEMONSTRATION.—Let $a^n - b^n$ be the difference of any powers of two quantities, and $a - b$ the difference of the same roots of those powers. Then is $a^n - b^n$ divisible by $a - b$.

Dividing until several remainders are obtained, we have

$$\begin{array}{rl|l}
 & a^n - \quad b^n & a - b \\
 & \underline{a^n - a^{n-1}b} & \overline{a^{n-1} + a^{n-2}b} \\
\text{1st Rem.} = & a^{n-1}b - b^n & \\
 & \underline{a^{n-1}b - a^{n-2}b^2} & \\
\text{2d Rem.} = & a^{n-2}b^2 - b^n & \\
n\text{th Rem.} = & a^{n-n}b^n - b^n & \\
= & a^0b^n - b^n & \\
= & b^n - b^n = 0. &
\end{array}$$

It is seen that the first term of the first remainder is $a^{n-1}b$; of the second remainder, $a^{n-2}b^2$; of the third remainder, $a^{n-3}b^3$; of the fourth remainder, $a^{n-4}b^4$. Hence the nth remainder will be $a^{n-n}b^n$; but $a^{n-n} = a^0 = 1$ (ART. 80, Prin. 4). Therefore, the first term of the nth remainder reduces to b^n.

Since the second term of the nth remainder is $-b^n$, the nth remainder is $b^n - b^n = 0$; that is, there is no remainder, and the division terminates. Therefore, $a^n - b^n$ is divisible by $a - b$.

EXAMPLES.

Factor—

1. $a^3 - b^3$.
2. $a^4 - b^4$.
3. $a^5 - b^5$.
4. $a^7 - b^7$.
5. $a^{11} - b^{11}$.
6. $x^6 - y^9$.
7. $x^9 - y^3$.
8. $x^5 - y^{10}$.
9. $x^{10} - y^{15}$.
10. $x^9 - y^{12}$.
11. $m^3 - 125$.
12. $n^5 - 32$.
13. $128 - a^7$.
14. $a^{11} - 1$.
15. $a^{3n} - b^{3m}$.

EXAMPLES.

Write by inspection the values of the following:

1. $(a+x)(a-x)$.
2. $(m+x)(m-x)$.
3. $(x+y)(x-y)$.
4. $(c+d)(c-d)$.
5. $(x+3)(x-3)$.
6. $(3-x)(3+x)$.
7. $(x-1)(x+1)$.
8. $(a+2b)(a-2b)$.
9. $(a-8x)(a+8x)$.
10. $(3x-7)(3x+7)$.
11. $(a^2-y^2)(a^2+y^2)$.
12. $(3m+4n)(3m-4n)$.
13. $(3a-5c)(3a+5c)$.
14. $(a^n+b^n)(a^n-b^n)$.
15. $(3v^2-4t^2)(3v^2+4t^2)$.
16. $(c^{3n}+d^{2n})(c^{3n}-d^{2n})$.
17. $(ax^n+bx^m)(ax^n-bx^m)$.
18. $(a^{m+n}+b^{m+n})(a^{m+n}-b^{m+n})$.
19. $(a^{n^2}-b^{m^2})(a^{n^2}+b^{m^2})$.
20. $(x^x+y^y)(x^x-y^y)$.

PRINCIPLE IV.

The product of two binomials having a common term equals the square of the common term, plus the algebraic sum of the other two terms into the common term, and the product of the unlike terms.

Thus, by multiplication,

$$\begin{array}{l} x+a \\ \underline{x+b} \\ x^2+ax \\ \underline{\quad\;\; +bx+ab} \\ x^2+(a+b)x+ab \end{array}$$

Also, $(m+x)(m+y)=m^2+(x+y)m+xy$.

And $(m+x)(m-y)=m^2+(x-y)m-xy$.

And $(m-x)(m-y)=m^2-(x+y)m+xy$.

And $(m+6)(m-2)=m^2+4m-12$.

EXAMPLES.

Write by inspection the values of the following:

1. $(x+1)(x+2)$.
2. $(x+4)(x-1)$.
3. $(x-4)(x+3)$.
4. $(x-2)(x-5)$.
5. $(m+5)(m-8)$.
6. $(x-7)(x+5)$.

7. $(a-8)(a+9)$.
8. $(x-5)(x-10)$.
9. $(a+5)(a-16)$.
10. $(n-10)(n+1)$.
11. $(3x-5)(3x+2)$.
12. $(a+b)(a+e)$.
13. $(x-a)(x+c)$.
14. $(x-b)(x-e)$.
15. $(2x+1)(2x+3)$.
16. $(x^2-4)(x^2-6)$.
17. $(7y-4)(7y-8)$.
18. $(4a-2b)(4a-3c)$.
19. $(a^n-b)(a^n-e)$.
20. $(3x^n-m^n)(3x^n+n^m)$.

GENERAL APPLICATION.

87. These principles may be applied with great convenience in finding the product of three or more binomials or the product of certain polynomials.

1. Expand $(a-b)(a+b)(a^2+b^2)$.

By Prin. III., $(a-b)(a+b)=a^2-b^2$.
Applying Prin. III. again, $(a^2-b^2)(a^2+b^2)=a^4-b^4$.
Hence, $(a-b)(a+b)(a^2+b^2)=a^4-b^4$.

2. Expand $(\overline{a+b}-c)(\overline{a+b}+c)$.

By Prin. III., this equals $(a+b)^2-c^2$.
By Prin. I., $(a+b)^2-c^2=a^2+2ab+b^2-c^2$.
Hence, $(a+b-c)(a+b+c)=a^2+2ab+b^2-c^2$.
Also, $(a-b-c)(a+b+c)=(a-\overline{b+c})(a+\overline{b+c})$
$=a^2-(b+c)^2=a^2-b^2-2bc-c^2$.

EXAMPLES.

Expand—

3. $(1-a)(1+a)(1+a^2)$.
4. $(x-y)(x+y)(x^2+y^2)$.
5. $(a+3)(a-3)(a^2+9)$.
6. $(m+n)(m-n)(m^2-n^2)$.
7. $(a+3)(a+3)(a+3)$.
8. $(a+b)(a+b)(a+b)$.

9. $(x - y + z)(x - y - z)$.
10. $(a^2 - ab + b^2)(a^2 + ab + b^2)$.
11. $(m^2 + mn + n^2)(m^2 - mn + n^2)$.
12. $(a^2 + a + 1)(a^2 - a - 1)$.
13. $(x^2 - xy + y^2)(x^2 + xy + y^2)$.
14. $(1 - n + n^2)(1 + n + n^2)$.
15. $(3a^2 + 3a - 1)(3a^2 - 3a + 1)$.
16. $(m - n)(m + n)(m^2 + n^2)(m^4 + n^4)$.
17. $(1 - a)(1 + a)(1 + a^2)(1 + a^4)(1 + a^8)$.
18. $(a - x)(a + x)(a^2 + x^2)(a^4 + x^4)(a^8 + x^8)$.
19. $(a + b + m + n)(a + b - m - n)$.
20. $(a + m - b - n)(a - m - b - n)$.
21. $(a^n - b^n)(a^n + b^n) a^{2n} + b^{2n})(a^{4n} + b^{4n})$.
22. $(a^n - y^n + b^n + x^n)(a^n - y^n - b^n - x^n)$.

FACTORING.

88. Factoring is the process of resolving a composite quantity into its factors.

The *Factors* of a composite quantity are the quantities which when multiplied together will produce it.

89. A **Prime Quantity** is one that cannot be produced by the multiplication of other quantities; as, 17, a, $a + b$, or $a^2 + b^2$.

In speaking of prime quantities in algebra we must exclude factoring by means of *fractional exponents* and *imaginary quantities;** otherwise many quantities regarded as prime could be factored.

For example, $a - b = (a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.

Or, $= (a^{\frac{1}{3}} - b^{\frac{1}{3}})(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}})$, etc.

$$a^2 + b^2 = (a + b\sqrt{-1})(a - b\sqrt{-1}).$$

---

* An imaginary quantity is the indicated even root of a negative quantity; as, $\sqrt{-a}$, or $\sqrt[2n]{-a}$.

90. By Art. 8, one of the two equal factors of a quantity is its *square root.*

Thus, $2a$ is the square root of $4a^2$.

I. TO FACTOR MONOMIALS.

91. The prime factors of monomials may be obtained by inspection, and therefore need no further consideration.

Thus, the prime factors of $6a^2b^3 = 2 \times 3aabbb$.

II. TO FACTOR BINOMIALS.

92. When the binomial is the difference of two squares.

1. Factor $a^2 - b^2$ and $4x^2 - 9y^2$.

Referring to Art. 86, Prin. III., we see that such binomials are the product of the sum and difference of two quantities.

$$\text{Hence} \quad a^2 - b^2 = (a + b)(a - b).$$
$$4x^2 - 9y^2 = (2x + 3y)(2x - 3y).$$

Therefore, to obtain the factors use the following

RULE.

Extract the square root of each term of the binomial, and make their sum one factor and their difference the other factor.

Note.—This is one of the most general and useful cases of factoring, and should always be used first when applicable.

EXAMPLES.

Factor—

2. $m^2 - n^2$.
3. $c^2 - d^2$.
4. $x^2 - y^2$.
5. $p^2 - q^2$.
6. $x^2 - 1$.
7. $4 - y^2$.
8. $x^2 - 4y^2$.
9. $4a^2 - 9b^2$.
10. $16m^2 - 25n^2$.
11. $36x^2 - 49a^2$.
12. $36l^2 - 64m^2$.
13. $x^2y^2 - a^2b^2$.
14. $m^4 - n^4$.
15. $a^8 - b^8$.
16. $a^4 - x^8$.
17. $m^{16} - n^{16}$.
18. $x^{16} - y^8$.
19. $a^{2n} - b^{2m}$.
20. $9a^{2n} - 16b^{2n}$.
21. $a^{14} - b^4$.
22. $x^{8n} - y^{8m}$.

23. $\frac{1}{4}x^2 - \frac{1}{9}y^2$.

24. $225x^2y^2 - 9$.

25. $289x^{2n} - 121y^{2n}$.

26. $81m^2 - 100n^6$.

27. $a^2b^6 - c^2d^8$.

28. $\frac{1}{16}x^4 - 1$.

29. $144 - a^2b^{12}$.

30. $a^{32} - b^{32}$.

31. $a^{16n} - b^{16n}$.

The same principle may be applied to the following binomial forms:

32. Factor $a^2 - (b + c)^2$.

$$\text{1st factor} = [a + (b + c)] = (a + b + c).$$
$$\text{2d factor} = [a - (b + c)] = (a - b - c).$$

33. Factor $(a + b)^2 - (c + d)^2$.

$$\text{1st factor} = [(a + b) + (c + d)] = (a + b + c + d).$$
$$\text{2d factor} = [(a + b) - (c + d)] = (a + b - c - d).$$

34. $(a + b)^2 - c^2$.

35. $(m + n)^2 - p^2$.

36. $(x + y)^2 - z^2$.

37. $m^2 - (x - y)^2$.

38. $(a - b)^2 - c^2$.

39. $a^2 - (c + x)^2$.

40. $4 - (a - b)^2$.

41. $(x - y)^2 - 9$.

42. $m^4 - (m^2 - 1)^2$.

43. $x^2 - (x + y)^2$.

44. $(3x - 5)^2 - 4x^2$.

45. $m^6 - (m^3 + 1)^2$.

46. $(x + y)^2 - (z + v)^2$.

47. $(m + n)^2 - (x - y)^2$.

48. $(a - x)^2 - (y + b)^2$.

49. $(m - n)^2 - (p - q)^2$.

50. $(x + y)^2 - (x - y)^2$.

51. $(a - b)^2 - (a + b)^2$.

52. $(2a - c)^2 - (3b - x)^2$.

53. $(2x - 4)^2 - (x - 4)^2$.

54. $(5x + 2y)^2 - (5 - 3y)^2$.

55. $(x - 1)^2 - (y + 1)^2$.

56. $(ax + by)^2 - (ax - by)^2$.

57. $(m^2 + n^2)^2 - (a^2 - b^2)^2$.

Polynomials can sometimes be arranged in the form of the difference of two squares, and factored like the preceding examples.

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This may be arranged and factored as follows:

$$x^2 + 2xy + y^2 - z^2 = (x + y)^2 - z^2 = (x + y + z)(x + y - z).$$

59. Factor $a^2 + b^2 - c^2 - d^2 - 2cd - 2ab$.

This may be factored thus:

$$
\begin{aligned}
&a^2 + b^2 - c^2 - d^2 - 2cd - 2ab \\
&= a^2 - 2ab + b^2 - c^2 - 2cd - d^2 \\
&= (a^2 - 2ab + b^2) - (c^2 + 2cd + d^2) \\
&= (a - b)^2 - (c + d)^2 \\
&= [(a - b) + (c + d)]\,[(a - b) - (c + d)] \\
&= (a - b + c + d)\,(a - b - c - d).
\end{aligned}
$$

60. $a^2 + 2ab + b^2 - c^2$.
61. $x^2 - 2xy + y^2 - z^2$.
62. $x^2 - a^2 - 2ab - b^2$.
63. $y^2 - m^2 + 2mn - n^2$.
64. $1 - m^2 - n^2 - 2mn$.
65. $2ab - a^2 - b^2 + 1$.
66. $a^2 - 2xy - x^2 - y^2$.
67. $a^2 - y^2 + b^2 - 2ab$.
68. $x^2 - 2yz + y^2 - z^2$.
69. $m^2 - 2mn + n^2 - p^2$.
70. $m^2 - n^2 + 2pn - p^2$.
71. $m^2 - n^2 - 2pn - p^2$.
72. $m^2 + n^2 - p^2 + 2mn$.
73. $x^2 - 1 + y^2 + 2xy$.
74. $a^2 + 2ab + b^2 - x^2 + 2xy - y^2$.
75. $m^2 - 2mn + n^2 - p^2 - 2pq - q^2$.
76. $a^2 - y^2 - b^2 + x^2 - 2ax + 2by$.
77. $a^2 - 9y^2 - 6xy + 4b^2 - 4ab - x^2$.
78. $a^2 - 9y^2 + 9x^2 - 4z^2 + 12yz + 6ax$.
79. $m^2 - 2xy + n^2 - x^2 + 2mn - y^2$.

Sometimes a quantity may be both *added to*, and *subtracted from*, a polynomial, and the terms then arranged in the form of the difference of two squares and factored like the preceding examples.

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Since $x^4 + x^2y^2 + y^4$ lacks x^2y^2 of being a perfect square, if x^2y^2 be added *to* the expression and then subtracted *from* it, the value will not be changed (ART. 54, Axiom 5).

Adding and subtracting x^2y^2, the expression becomes

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2 \\ &= [(x^2 + y^2) + xy]\,[(x^2 + y^2) - xy] \\ &= (x^2 + xy + y^2)\,(x^2 - xy + y^2). \end{aligned}$$

81. Factor $a^4 + a^2 + 1$.

Adding and subtracting a^2,

$$\begin{aligned} a^4 + a^2 + 1 &= a^4 + 2a^2 + 1 - a^2 \\ &= (a^2 + 1)^2 - a^2 \\ &= (a^2 + a + 1)\,(a^2 - a + 1). \end{aligned}$$

82. $m^4 + m^2n^2 + n^4$.
83. $1 + y^2 + y^4$.
84. $b^4 + b^2c^2 + c^4$.
85. $a^4 + a^2x^2 + x^4$.
86. $x^4 - 3x^2y^2 + y^4$.
87. $a^4 - 6a^2n^2 + n^4$.
88. $a^4 - 7a^2y^2 + y^4$.
89. $m^4 - 27m^2n^2 + n^4$.
90. $l^4 - 38l^2m^2 + m^4$.
91. $4x^4 - 16x^2y^2 + 9y^4$.
92. $9a^4 + 14a^2b^2 + 25b^4$.
93. $25a^4 + a^2x^2 + x^4$.
94. $25x^4 - 79x^2y^2 + 49y^4$.
95. $25a^4 - 41a^2b^2 + 16b^4$.
96. $4x^4 - 37x^2y^2 + 9y^4$.
97. $49m^4 - 11m^2n^2 + 25n^4$.
98. $36x^4 - 16x^2y^2 + 49y^4$.
99. $81a^4 + 20a^2b^2 + 4b^4$.
100. $25a^4 - 49a^2y^2 + 16y^4$.
101. $4x^4 - 28x^2a^2 + 9a^4$.

102. $a^4 + 4$.
103. $x^4 + 64$.
104. $a^4 + 4x^4$.
105. $4a^4 + 16x^4$.
106. $x^8 + 324$.
107. $a^4b^4 + 4c^4$.
108. $x^8 + 1024$.
109. $x^{12} + 2500$.
110. $a^8x^4 + 4b^4$.

93. A **Theorem** is a truth to be demonstrated. A *Demonstration* is a course of reasoning employed in proving a theorem.

94. When the binomial is the sum or difference of any powers of two quantities.

This case relates to binomials of the form of $a^3 - b^3$, $a^3 + b^3$, $a^4 - b^4$, $a^6 + b^6$, $a^9 + b^6$, $a^3 + b^{12}$, etc.

THEOREM I.

The difference of any powers of two quantities is divisible by the **difference** *of the* **same** *roots of those powers.*

ILLUSTRATION.

$$\begin{aligned}(a^3 - b^3) &\div (a - b) = a^2 + ab + b^2.\\ (a^9 - b^6) &\div (a^3 - b^2) = a^6 + a^3b^2 + b^4.\\ (a^4 - b^{12}) &\div (a - b^3) = a^3 + a^2b^3 + ab^6 + b^9.\end{aligned}$$

DEMONSTRATION.—Let $a^n - b^n$ be the difference of any powers of two quantities, and $a - b$ the difference of the same roots of those powers. Then is $a^n - b^n$ divisible by $a - b$.

Dividing until several remainders are obtained, we have

$$\begin{array}{rl|l} & a^n - \qquad b^n & a - b \\ \cline{3-3} & \underline{a^n - a^{n-1}b} & a^{n-1} + a^{n-2}b \\ \cline{2-3} \text{1st Rem.} = & a^{n-1}b - b^n & \\ & \underline{a^{n-1}b - a^{n-2}b^2} & \\ \text{2d Rem.} = & a^{n-2}b^2 - b^n & \\ n\text{th Rem.} = & a^{n-n}b^n - b^n & \\ = & a^0b^n - b^n & \\ = & b^n - b^n = 0. & \end{array}$$

It is seen that the first term of the first remainder is $a^{n-1}b$; of the second remainder, $a^{n-2}b^2$; of the third remainder, $a^{n-3}b^3$; of the fourth remainder, $a^{n-4}b^4$. Hence the nth remainder will be $a^{n-n}b^n$; but $a^{n-n} = a^0 = 1$ (ART. 80, Prin. 4). Therefore, the first term of the nth remainder reduces to b^n.

Since the second term of the nth remainder is $-b^n$, the nth remainder is $b^n - b^n = 0$; that is, there is no remainder, and the division terminates. Therefore, $a^n - b^n$ is divisible by $a - b$.

EXAMPLES.

Factor—

1. $a^3 - b^3$.
2. $a^4 - b^4$.
3. $a^5 - b^5$.
4. $a^7 - b^7$.
5. $a^{11} - b^{11}$.
6. $x^6 - y^9$.
7. $x^9 - y^3$.
8. $x^5 - y^{10}$.
9. $x^{10} - y^{15}$.
10. $x^9 - y^{12}$.
11. $m^3 - 125$.
12. $n^5 - 32$.
13. $128 - a^7$.
14. $a^{11} - 1$.
15. $a^{3n} - b^{3m}$.

THEOREM II.

The difference of any powers of two quantities is divisible by the **sum** *of the same* **even** *roots of those powers.*

ILLUSTRATION.

$$(a^2 - b^2) \div (a + b) = a - b.$$
$$(a^8 - b^4) \div (a^2 + b) = a^6 - a^4b + a^2b^2 - b^3.$$
$$(a^4 - b^{12}) \div (a + b^3) = a^3 - a^2b^3 + ab^6 - b^9.$$

DEMONSTRATION.—Let $a^n - b^n$ be the difference of any powers of two quantities, and $a + b$ the sum of the same even roots of those powers. Then is $a^n - b^n$ divisible by $a + b$.

Dividing until four remainders are obtained, we have

$$\begin{array}{l|l}
a^n - \quad b^n & a + b \\
\underline{a^n + a^{n-1}b} & \overline{a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3}
\end{array}$$

$$\begin{array}{rl}
\text{1st Rem.} = & - a^{n-1}b - b^n \\
& \underline{- a^{n-1}b - a^{n-2}b^2} \\
\text{2d Rem.} = & + a^{n-2}b^2 - b^n \\
& \underline{+ a^{n-2}b^2 + a^{n-3}b^3} \\
\text{3d Rem.} = & - a^{n-3}b^3 - b^n \\
& \underline{- a^{n-3}b^3 - a^{n-4}b^4} \\
\text{4th Rem.} = & + a^{n-4}b^4 - b^n \\
\text{If } n \text{ is } \textit{even}\text{, the } n\text{th Rem.} = & + a^{n-n}b^n - b^n \\
= & + a^0b^n - b^n \\
= & b^n - b^n = 0.
\end{array}$$

It is seen that the first term of the odd remainders is $-$, and of the even remainders is $+$. Hence if n is even, the first term of the nth remainder is $+ a^{n-n}b^n$; but $a^{n-n} = a^0 = 1$ (ART. 80, Prin. 4). Therefore, the first term of the nth remainder reduces to b^n.

Since the second term of the nth remainder is $- b^n$, the nth remainder is $b^n - b^n = 0$; that is, there is no remainder, and the division terminates. Therefore, $a^n - b^n$ is divisible by $a + b$ if n is even.

EXAMPLES.

Factor—

1. $a^2 - b^4$.
2. $a^4 - b^4$.
3. $x^6 - b^6$.
4. $a^8 - b^8$.
5. $m^{10} - n^{10}$.
6. $16 - a^4$.
7. $a^4 - 1$.
8. $x^8 - 1$.
9. $a^4 - b^{16}$.
10. $a^4b^8 - 1$.
11. $81 - y^4$.
12. $a^6x^6 - y^{12}$.
13. $x^{10} - y^{20}$.
14. $a^{4n} - b^{4n}$.
15. $x^{8n} - y^{4n}$.

THEOREM III.

The sum of any powers of two quantities is divisible by the **sum** *of the same* **odd** *roots of those powers.*

ILLUSTRATION.

$$\begin{aligned}(a^3+b^3) &\div (a+b) = a^2 - ab + b^2.\\ (a^5+b^{10}) &\div (a+b^2) = a^4 - a^3b^2 + a^2b^4 - ab^6 + b^8.\\ (a^6+b^6) &\div (a^2+b^2) = a^4 - a^2b^2 + b^4.\end{aligned}$$

DEMONSTRATION.—Let $a^n + b^n$ be the sum of any two powers of two quantities, and $a + b$ the sum of the same odd roots of those powers. Then is $a^n + b^n$ divisible by $a + b$.

Dividing until four remainders are obtained, we have

$$\begin{array}{l|l} a^n + \qquad b^n & a+b \\ \underline{a^n + a^{n-1}b} & \underline{a^{n-1} - a^{n-2}b + a^{n-3}b^2 - a^{n-4}b^3} \end{array}$$

$$\begin{aligned}
\text{1st Rem.} &= -a^{n-1}b + b^n\\
&\quad \underline{-a^{n-1}b - a^{n-2}b^2}\\
\text{2d Rem.} &= a^{n-2}b^2 + b^n\\
&\quad \underline{+a^{n-2}b^2 + a^{n-3}b^3}\\
\text{3d Rem.} &= -a^{n-3}b^3 + b^n\\
&\quad \underline{-a^{n-3}b^3 - a^{n-4}b^4}\\
\text{4th Rem.} &= a^{n-4}b^4 + b^n\\
\text{If } n \text{ is } \textit{odd}\text{, the } n\text{th Rem.} &= -a^{n-n}b^n + b^n\\
&= -a^0b^n + b^n\\
&= -b^n + b^n = 0.
\end{aligned}$$

It is seen that the first term of the odd remainders is $-$, and of the even remainders is $+$. Hence if n is odd the first term of the nth remainder is $-a^{n-n}b^n$; but $a^{n-n} = a^0 = 1$ (ART. 80, Prin. 4). Therefore, the first term of the nth remainder reduces to $-b^n$.

Since the second term of the nth remainder is $+ b^n$, the nth remainder is $-b^n + b^n = 0$; that is, there is no remainder, and the division terminates. Therefore, $a^n + b^n$ is divisible by $a + b$ if n is odd.

EXAMPLES.

Factor—

1. $a^5 + x^5$.
2. $a^7 + y^7$.
3. $a^9 + b^9$
4. $a^6 + n^6$.
5. $n^{10} + m^{10}$.
6. $x^{12} + y^{12}$.
7. $a^{14} + y^{14}$.
8. $x^{15} + y^{10}$.
9. $a^{14} + x^7$.
10. $m^6 + n^9$.
11. $a^{15} + n^6$.
12. $n^3 + 27$.
13. $a^5 + b^{10}c^5$.
14. $1 + n^7$.
15. $a^{3n} + b^{12n}$.

MISCELLANEOUS EXAMPLES.

16. $a^3 - 343$.
17. $32x^5 - y^5$.
18. $a^5 + x^{10}y^{10}$.
19. $a^8 - b^8$.
20. $a^{12} - b^{12}$.
21. $x^{16} - y^{16}$.
22. $a^{10}y^5 + x^5b^{10}$.
23. $a^5b^{10} - x^{10}y^5$.
24. $a^6 - 1$.
25. $x^{14} - y^7$.
26. $x^4 - y^4$.
27. $a^5b^5 - x^{15}y^{15}$.
28. $x^7 + y^{21}$.
29. $a^4 - b^{16}$.
30. $a^9 - b^3$.

III. TO FACTOR TRINOMIALS.

95. A **Quadratic Trinomial** is one whose form is $x^2 \pm ax \pm b$, in which b is the product of two quantities and a their algebraic sum.

96. When the trinomial is quadratic.

A quadratic trinomial will assume one of the six following forms:

First Form.—Factor $a^2 + 2ab + b^2$.

By Art. 86, Prin. I., this is evidently the square of the sum of a and b. Hence,

$$a^2 + 2ab + b^2 = (a + b)(a + b) = (a + b)^2.$$

Second Form.—Factor $a^2 - 2ab + b^2$.

By Prin. II., Art. 86, this is evidently the square of the difference of a and b. Hence,

$$a^2 - 2ab + b^2 = (a - b)(a - b) = (a - b)^2.$$

Third Form.—Factor $a^2 + 7a + 12$.

By Prin. IV., Art. 86, the first term of each binomial is a, the product of the other two terms is 12, and their sum is 7. The factors of 12 are

$$1 \times 12,$$
$$2 \times 6,$$
$$3 \times 4.$$

Since the sum of the two terms is 7, they must be 3 and 4. Hence,

$$a^2 + 7a + 12 = (a + 3)(a + 4).$$

Fourth Form.—Factor $a^2 - 7a + 12$.

This form is factored like the Third Form, with the second term in each factor *negative*. Hence,

$$a^2 - 7a + 12 = (a - 3)(a - 4).$$

Fifth Form.—Factor $a^2 + a - 12$.

This form is factored like the Third Form, with the second term in one factor *positive* and the other *negative*, the positive term being the greater. Hence,

$$a^2 + a - 12 = (a - 3)(a + 4).$$

Sixth Form.—Factor $a^2 - a - 12$.

This form is factored like the Third Form, with the second term in one factor *positive* and the other *negative*, the negative term being the greater. Hence,

$$a^2 - a - 12 = (a + 3)(a - 4).$$

EXAMPLES.

Factor—

1. $m^2 + 2mn + n^2$.
2. $a^2 - 8a + 16$.
3. $x^2 + 3x + 2$.
4. $a^2 - 4a + 3$.
5. $a^2 + a - 2$.
6. $n^4 - n^2 - 2$.
7. $x^2 + 12x + 36$.
8. $m^8 - 34m^4 + 289$.
9. $y^2 + 6y + 5$.
10. $n^2 - 12n + 11$.
11. $x^2 + x - 6$.
12. $n^2 - n - 20$.
13. $a^2 + 8a + 16$.
14. $s^2 - 42s + 441$.
15. $m^2 + 5m + 6$.
16. $x^2 - 16x + 15$.
17. $m^2 + m - 12$.
18. $x^2 - x - 42$.
19. $y^2 + 10y + 25$.
20. $1 - 44a^2 + 484a^4$.
21. $x^2 + 13x + 22$.
22. $z^2 - 12z + 20$.
23. $n^2 + n - 20$.
24. $z^4 - 13z^2 - 140$.
25. $a^2 + 4a + 4$.
26. $a^2 - 16ab + 64b^2$.
27. $n^2 + 43n + 82$.
28. $x^8 - 11x^4 + 28$.

29. $y^2 + y - 132$.

30. $x^2 - 21x - 100$.

31. $x^2 + 12xy + 36y^2$.

32. $m^2 - 14mn + 49n^2$.

33. $x^6 - 23x^3 + 120$.

34. $x^2 - 24x + 63$.

35. $n^2 + 9n - 36$.

36. $r^2 - 48r - 100$.

37. $a^2 + 24ab + 144b^2$.

38. $x^2 - 16xy + 64y^2$.

39. $a^2 + 19a + 84$.

40. $n^4 - 15n^2 + 50$.

41. $y^4 + 5y^2 - 24$.

42. $x^2 - 2x - 80$.

43. $m^2 + 8mn + 16n^2$.

44. $m^2 - 32m + 256$.

45. $y^4 + 21y^2 + 110$.

46. $a^4 - 30a^2b + 81b^2$.

47. $a^2 + 9a - 10$.

48. $n^2 - 17n - 200$.

49. $1 + 20n^2 + 100n^4$.

50. $1 - 36a^2 + 324a^4$.

51. $m^2 + 6mn + 9n^2$.

52. $x^2 - xy + \frac{1}{4}y^2$.

53. $a^2 + 18a + 17$.

54. $x^2 - 29x + 168$.

55. $x^2 + 6x - 135$.

56. $a^2 - 6ax - 55x^2$.

57. $a^2 + 10a + 24$.

58. $m^2 - 26m + 105$.

59. $a^2 + 15a - 100$.

60. $x^4 - 25x^2 - 150$.

61. $x^2 + 38x + 361$.

62. $m^2n^2 - 60mn + 900$.

63. $n^2 + 11n + 24$.

64. $a^2 - 44a + 160$.

65. $a^4 + 3a^2 - 54$.

66. $a^4 - 31a^2 - 360$.

67. $y^2 + 42y + 441$.

68. $x^2y^4 - 80xy^2 + 1600$.

69. $x^2 + 14x + 24$.

70. $y^2 - 36y + 99$.

71. $a^2 + 3a - 40$.

72. $n^2 - 39n - 270$.

73. $x^{2n} + 26x^n + 169$.

74. $y^{4n} - 22y^{2n} + 121$.

75. $a^{2n} + 28a^n + 75$.

76. $x^4 - 23x^2 + 60$.

77. $n^4 + 34n^2m - 72m^2$.

78. $m^{4n} - 45m^{2n} - 364$.

79. $a^{2n} - 2a^nn^2 + n^4$.

80. $x^2 + 44x + 484$.

81. $x^2 + 53x + 360$.

82. $x^6 - 37x^3 - 360$.

83. $n^2 - n - 156$.

84. $x^{2x} + 6x^x - 91$.

97. When the trinomial is of the form $ax^2 \pm bx \pm c$.

1. Factor $2x^2 + 7xy - 15y^2$.

It is evident that the first terms of the two factors are

$$(2x \quad)(x \quad).$$

Since the third term is $-15y^2$, the other two terms of the binomials are either $-3y$ and $5y$, or $3y$ and $-5y$. These factors must be selected and arranged to produce the middle term. They are found by trial to be $-3y$ and $5y$. Hence,

$$2x^2 + 7xy - 15y^2 = (2x - 3y)(x + 5y).$$

NOTE.—This case is factored largely *by trial*, but a little practice will enable the student to factor trinomials of this form with ease.

EXAMPLES.

Factor—

1. $6x^2 + 29x + 35$.
2. $5x^2 + 37x + 14$.
3. $14x^2 + 25x + 9$.
4. $8x^2 + 22x + 15$.
5. $15a^2 + 39a + 24$.
6. $2a^2 + 5a - 3$.
7. $20x^2 - 23x + 6$.
8. $6x^2 + 19x - 20$.
9. $21x^2 - x - 10$.
10. $27x^2 + 9x - 6$.
11. $63x^2 + 130x + 63$.
12. $44x^2 - 75x - 54$.
13. $15x^2 + 31xy + 14y^2$.
14. $24a^2 + 33ab - 30b^2$.
15. $12n^2 - 29mn + 15m^2$.
16. $6x^2 + 7xy + 2y^2$.
17. $22x^2 + 27xy - 9y^2$.
18. $12x^2 - 32xy + 21y^2$.
19. $4x^2 + 7xy - 15y^2$.
20. $5x^2 - 29xy + 36y^2$.
21. $8x^2 + 8xy - 48y^2$.
22. $55x^2 + 79xy - 36y^2$.
23. $20a^2 + 13ab - 21b^2$.
24. $143a^2 - 22ab - 165b^2$.
25. $70n^2 - 221nm + 70m^2$.
26. $33x^2 - 34xy - 35y^2$.
27. $35x^2 - xy - 12y^2$.
28. $55x^2 - 96xy - 55y^2$.
29. $40x^2 + 3xy - 28y^2$.
30. $12a^2 - 70ay - 12y^2$.

IV. TO FACTOR POLYNOMIALS.

98. When the polynomial contains a common factor.

1. Factor $5a^2c - 10abc + 15ac^2 - 20ac^3$.

By examining the polynomial we see that $5ac$ is a factor common to all the terms. Dividing by $5ac$, we obtain the other factor. Hence

$$5ac\overline{)5a^2c - 10abc + 15ac^2 - 20ac^3}.$$
$$a - 2b + 3c - 4c^2$$

EXAMPLES.

Factor—

2. $6an^2b - 21nb^2$.
3. $20x^3y + 36xy^3$.
4. $72a^2x - 90abx$.
5. $4a^3bx + 4b^4x$.
6. $a^4 - 16a^2$.
7. $x^5 - 16x$.
8. $2a^2 + 4ab + 2b^2$.
9. $7x^6 - 7x^8$.
10. $3x^7y + 3xy^8$.
11. $3x^7y + 3xy^{10}$.
12. $2x^3 - 8x^2 - 6x$.
13. $5y^4 + 45y^3 + 100y^2$.
14. $70x^3 + 77x^2 - 42x$.
15. $a^5 + a^3 + a$.
16. $5m^7 - 5mn^{12}$.
17. $3x^7 - 3xy^{6n}$.
18. $c(a + b)^2 - c^3$.
19. $2a^3 - 2ab^2 - 4abc - 2ac^2$.
20. $x^5y + x^3y^3 + xy^5$.
21. $y + y^3 + y^4$.
22. $5a^{10} + 5ax^9$.
23. $a^3 + 2a^2b + ab^2 - ac^2$.

99. When the terms can be grouped.

1. Factor $ax - by - bx + ay$.

It is evident that

$$\begin{aligned} ax - by - bx + ay &= ax + ay - bx - by \\ &= a(x + y) - b(x + y) \\ &= (a - b)(x + y), \textit{ Ans.} \end{aligned}$$

2. Factor $3x^3 - 6x^2y - 2x + 4y$.

$$\begin{aligned} 3x^3 - 6x^2y - 2x + 4y &= (3x^3 - 6x^2y) - (2x - 4y) \\ &= 3x^2(x - 2y) - 2(x - 2y) \\ &= (3x^2 - 2)(x - 2y), \textit{ Ans.} \end{aligned}$$

3. Factor $x^3 - 29x + 20$.

By separating and grouping,

$$\begin{aligned} x^3 - 29x + 20 &= x^3 - 25x - 4x + 20 \\ &= x(x^2 - 25) - 4(x - 5) \\ &= x(x - 5)(x + 5) - 4(x - 5) \\ &= (x - 5)(x^2 + 5x - 4), \textit{ Ans.} \end{aligned}$$

4. Factor $x^3 + 3x + 14$.

$$\begin{aligned} x^3 + 3x + 14 &= x^3 - 4x + 7x + 14 \\ &= x(x^2 - 4) + 7(x + 2) \\ &= (x + 2)(x^2 - 2x + 7), \textit{ Ans.} \end{aligned}$$

5. Factor $y^3 - 5y^2 + 12$.

$$\begin{aligned} y^3 - 5y^2 + 12 &= y^3 - 2y^2 - 3y^2 + 12 \\ &= y^2(y - 2) - 3(y^2 - 4) \\ &= (y - 2)(y^2 - 3y - 6), \textit{ Ans.} \end{aligned}$$

EXAMPLES.

Factor—

6. $4ax - 12x + 2a - 6$.
7. $ax + bx + ac + bc$.
8. $3ac - 3c - 4ab + 4b$.
9. $2cm - 2cn - 3nx - 3mx$.
10. $3a^2 - 4ax - 3ax + 4x^2$.
11. $a^2 - ab - 2(a - b)$.
12. $bx - cx + bc - x^2$.
13. $x^3 + y^3 + x + y$.
14. $x^3 - 52x - 21$.
15. $x^3 - 31x + 30$.
16. $y^3 - 8y^2 + 45$.
17. $y^3 - 80y + 200$.
18. $a^3 - 10a^2 + 147$.
19. $y^3 - 8y + 32$.
20. $x^3 - 19x - 12$.
21. $x^3 - 32x - 35$.
22. $n^3 - 9n^2 + 98$.
23. $x^3 - y^3 - x + y$.

100. Suggestions.—The student should be careful to select the best method of factoring an expression.

Thus, there are several methods of resolving $a^6 - b^6$ into its prime factors; but the best method is as follows:

$$\begin{aligned} a^6 - b^6 &= (a^3 + b^3)(a^3 - b^3) \\ &= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2). \end{aligned}$$

Also, to factor $a^8 - b^8$.

$$\begin{aligned} a^8 - b^8 &= (a^4 + b^4)(a^4 - b^4) \\ &= (a^4 + b^4)(a^2 + b^2)(a^2 - b^2) \\ &= (a^4 + b^4)(a^2 + b^2)(a + b)(a - b). \end{aligned}$$

The following order of factoring is usually the best:

1st. Remove any monomial factor that the expression may contain (Art. 98).

2d. If the expression is a binomial, apply the principle of Art. 92; then the principles of Art. 94.

MISCELLANEOUS EXAMPLES.

Factor—

1. $4a^4 - 4ab^3$.
2. $x^{12} - x^6$.
3. $y^5 - y$.
4. $a^{13} - ay^{12}$.
5. $2a^3 - 2ab^2$.
6. $2x^2 - 4x - 126$.
7. $3(m - n)^2 - 12$.
8. $25a^4 - 19a^2 + 1$.
9. $4x^4 - 12x^3y + 9x^2y^2$.
10. $m^{12} + m^9$.
11. $36a^2 - 12a + 1$.
12. $6y^2 + ay - 35a^2$.
13. $x^{16} - y^{16}$.
14. $a^{24} + b^{24}$.
15. $z^4 - 6z^2y^2 + y^4$.
16. $y^2 - 3y - 130$.
17. $x^2 - 26x - 120$.
18. $(a - b)^4 + (a - b)^2 + 1$.
19. $x^5 + x^3 + x$.
20. $x^3 + x^2 + x + 1$.
21. $2a^2 + 7ab - 72b^2$.
22. $x^{5n} - y^{10n}$.
23. $a^2 + 2a - 99$.
24. $x^4 - 3x^2y^2 + y^4$.

25. $8a^2 - 22ay + 15y^2$.

26. $x^3 + 6x^2 + 11x + 6$.

27. $a^{15} + y^{12}$.

28. $m^2 + 17m - 110$.

29. $x^{2n} + 32x^n + 256$.

30. $y^6 + y^4 + y^2$.

31. $a^2 - 4ab + 4b^2 + 5a - 10b$.

32. $(a - b)^2 - d^2$.

33. $x^2 - (a - b)^2$.

34. $(x - y)^2 - (x + y)^2$.

35. $x^3 + 4x^2 - 17x - 60$.

36. $a^4 + 4b^4$.

37. $x^6 - 10x^3 - 200$.

38. $4ax^7 + 4xa^7$.

39. $m^2 - 7m - 120$.

40. $x^3 - 39x - 70$.

41. $y^{12} + 3y^6 - 340$.

42. $6a^2 - 13ax + 6x^2$.

43. $x^2 + 3x - 154$.

44. $49a^{2n} + 126a^nb^n + 81b^{2n}$.

45. $4a^2 - 81b^2$.

46. $x^3 + x^2 - 150$.

47. $x^3 + 6x - 20$.

48. $c^2 - c - 12$.

49. $m^{18} - n^9$.

50. $400x^2 - (2x - y)^2$.

51. $n - n^7$.

52. $1 - x^2 - y^2 + 2xy$.

53. $x^2 + 2x(a + b) + (a + b)^2$.

54. $x^{14} + y^{14}$.

55. $a^3 - 28a - 48$.

56. $z^2 - z - 132$.

57. $2ab - a^2 - b^2 + c^2 + d^2 - 2cd$.

58. $x^3 + 3x^2 + 3x + 1$.

59. $a^4 + 4$.

60. $x^{2n} + 2x^n + 1$.

61. $(a + b)^2 - x^2 - 2xy - y^2$.

62. $64 + x^4$.

63. $24x^4 + 5x^2y^2 - 36y^4$.

64. $(1 - a)^2 - (1 - b)^2$.

65. $x^2 - 2xy + y^2 - 6x + 6y$.

66. $x^{12} - y^{28}$.

67. $c^{24} - d^{24}$.

68. $y^3 - 49y + 120$.

69. $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.

70. $(m^2 - n^2 - p^2)^2 - 4n^2p^2$.

71. $x^3 - 12x^2 + 243$.

72. $x^3 - 28x - 48$.

73. $100a^2 - (p + q).^2$

74. $a^3 - 3a^2b + 3ab^2 - b^3$.

75. $a^4 + b^4$.

76. $x^3 - 10x^2 + 31x - 30$.

77. $(a^2 - b^2)^2 - (c^2 - d^2)^2$.

78. $(m - n)^2 - p^2$.

79. $(x^2 + x - 4)^2 - 4$.

80. $x^3 - x^2 - x + 1$.

HIGHEST COMMON DIVISOR.

101. A **Divisor*** of a quantity is a quantity that will exactly divide it.

Thus, $3a$ is a divisor of $6a^2$.

102. A **Common Divisor** of two or more quantities is a quantity that will exactly divide each of them.

Thus, $3a$ is a common divisor of $6a^2$ and $9a^2b$.

103. The **Highest Common Divisor** (H. C. D.) of two or more quantities is a divisor of the highest degree which will exactly divide them.

Thus, $3a^2$ is the H. C. D. of $6a^2$ and $9a^2b$.

PRINCIPLE.

The highest common divisor of two or more quantities is the product of all their common prime factors.

CASE I.

104. When the quantities are readily factored.

1. Find the highest common divisor of $6a^2b^2x$ and $9a^2bx^2$.

PROCESS.

$$\begin{array}{lrl} & 6a^2b^2x & = 2 \times 3 \times aa \times bb \times x \\ & \underline{9a^2bx^2} & \underline{= 3 \times 3 \times aa \times b \times xx} \\ \text{H. C. D.} & & = 3 \times aa \times b \times x = 3a^2bx. \end{array}$$

Since the highest common divisor is the product of all the common prime factors (ART. 103, Prin.), we resolve the quantities into their prime factors. The only prime factors common to both quantities are 3, aa, b, and x, and their product, $3a^2bx$, is the highest common divisor.

2. Find the H. C. D. of $a^4 - b^4$ and $a^6 + b^6$.

PROCESS.

$$\begin{array}{lrl} & a^4 - b^4 & = (a^2 + b^2)(a + b)(a - b) \\ & \underline{a^6 + b^6} & \underline{= (a^2 + b^2)(a^4 - a^2b^2 + b^4)} \\ \text{H. C. D.} & & = a^2 + b^2. \end{array}$$

---

* This technical idea of *divisor* must not be confused with the general idea of divisor in ART. 79.

Therefore, to find the highest common divisor when the quantities are readily factored,

RULE.

Resolve the quantities into their prime factors, and take the product of all the common factors.

EXAMPLES.

Find the H. C. D. of—

3. $9x^2y^3z$ and $12xy^2z^3$.
4. $12m^2n^3y^5$ and $16m^3n^2y^3$.
5. $14l^2m^4n^6$ and $21l^6m^4n^2$.
6. $15xy^2z^3v^4$ and $20x^4y^3z^2v$.
7. $25x^4n^8m^{2x}$ and $30x^2n^7m^x$.
8. $30x^{2n}y^{3n}z^{4n}$ and $40x^{2n}y^{4n}z^{2n}$.
9. $5x^2y^3z^4$, $6xy^2z^3$, and $7x^3y^2z$.
10. $9m^2n^3$, $12m^3n^2$, and $15m^2n^2$.
11. $12a^3x^2y^4z^6$, $18a^2x^3y^2z^4$, and $24a^3x^4y^5z^5$.
12. $6m^2x^3y$, $12m^3x^5y^2$, and $20m^2x^3y^4$.
13. $x^2 - y^2$ and $x^2 + 2xy + y^2$.
14. $x^2 - 2x$, $x^2 - 4$, and $(x - 2)^2$.
15. $z^2 + 3z$, $z^2 - 9$, and $(z + 3)^2$.
16. $m^2 - m$, $m^2 - 1$, and $m^2 - 2m + 1$.
17. $x^2 + x - 6$ and $x^2 + 3x - 10$.
18. $x^2 + 10x + 21$ and $x^2 - 2x - 15$.
19. $x^2 + 2x - 63$ and $x^2 - 13x + 42$.
20. $x^2 + x - 132$ and $x^2 - x - 110$.
21. $x^2 - 81$ and $x^2 - x - 90$.
22. $6x^2 - x - 12$ and $10x^2 - 9x - 9$.
23. $x^3 + y^3$ and $x^2 + 2xy + y^2$.

24. $x^6 + y^6$ and $x^{10} + y^{10}$.

25. $x^6 - y^6$ and $x^9 - y^9$.

26. $x^6 + y^6$ and $x^4 - y^4$.

27. $x^2 - y^2$, $(x - y)^2$, and $x^2y - xy^2$.

28. $x^2 - 6x + 5$, $x^2 - 7x + 10$, and $x^2 - 8x + 15$.

29. $x^3 - x^2 - 12x$ and $x^2 - 4x - 21$.

30. $x^6 + y^6$, $x^4 - y^4$, and $x^4 + 2x^2y^2 + y^4$.

31. $x^4 + x^2y^2 + y^4$ and $x^3y - x^2y^2 + xy^3$.

32. $x^4 + x^2 + 1$ and $x^3 + 1$.

33. $x^{12} + y^{12}$ and $x^8 - y^8$.

34. $(a + b)^2 - c^2$ and $a^2 - (b + c)^2$.

35. $a^2 - (b - c)^2$ and $b^2 - (a - c)^2$.

36. $x^2 - (a + b + c)^2$ and $(x - a)^2 - (b + c)^2$.

37. $3a^2 + 2a - 1$ and $a^3 + a^2 + a + 1$.

38. $x^2 - y^2 - 2yz - z^2$ and $x^2 + 2xy + y^2 - z^2$.

39. $x^6 + y^6$, $x^4 - x^2y^2 + y^4$, and $x^8 + x^4y^4 + y^8$.

40. $(x^3 + y^3)(x^2 + xy + y^2)$ and $(x^3 - y^3)(x^2 - xy + y^2)$.

CASE II.

105. When the quantities are not readily factored.

PRINCIPLE I.

A divisor of a quantity is a divisor of any number of times that quantity.

This is axiomatic, since if a is contained in b, n times, it is contained in $2b$ twice as many times, in $3b$ three times as many times, etc.

PRINCIPLE II.

A common divisor of two quantities is a divisor of their sum, and also of their difference.

c^2, which is a common divisor of ac^2 and bc^2, is also a divisor of $ac^2 + bc^2$ or $(a + b)c^2$, and of $ac^2 - bc^2$ or $(a - b)c^2$.

PRINCIPLE III.

Either of two quantities may be multiplied or divided by a factor not found in the other, without changing their highest common divisor.

This is evident from ART. 103, Prin.

106. In Arithmetic we find the H. C. D. as follows:

Divide the greater number by the less; if there is a remainder, divide the divisor by it, and so on, until there is no remainder. The last divisor will be the highest common divisor.

ILLUSTRATION.—Find the H. C. D. of 36 and 63.

$$\begin{array}{l}
36)63(1\\
\quad\ \underline{36}\\
\quad\ 27)36(1\\
\qquad\quad \underline{27}\\
\qquad\qquad 9)27(3\\
\qquad\qquad\quad \underline{27}
\end{array}$$

Hence, 9 is the H. C. D. of 36 and 63.

107. We will now prove that the same principle is true for any two algebraic quantities.

ILLUSTRATION.—Find the H. C. D. of A and B.

Suppose A is contained in B, q times, with a remainder of R; R is contained in A, q' times, with a remainder of R'; and R' is contained in R, q'' times, with no remainder. Then R' is the H. C. D. of A and B.

$$\begin{array}{l}
A)\,B\,(q\\
\quad \underline{qA}\\
\quad\ R)\ A\ (q'\\
\qquad\quad \underline{q'R}\\
\qquad\qquad R')\ R\ (q''\\
\qquad\qquad\quad\ \underline{q''R'}\\
\qquad\qquad\qquad\ 0
\end{array}$$

A and B are each a number of times the H. C. D.; then $B - qA$, or R, is a number of times the H. C. D. (ART. 105, Prin. 2).

Now, since A and R are each a number of times the H. C. D. and any divisor of R and qA is a divisor of B (ART. 105, Prin. 2), the question is reduced to finding the H. C. D. of A and R.

In like manner we can reduce it to finding the H. C. D. of R and R'; but, by hypothesis, this is R'. Hence R' is the H. C. D. of A and B.

1. Find the H. C. D. of $3x^2 + x - 2$ and $6x^3 + 17x^2 + 4x - 12$.

$$\begin{array}{l}
3x^2 + x - 2)6x^3 + 17x^2 + 4x - 12(2x + 5 \\
\qquad\qquad\quad \underline{6x^3 + 2x^2 - 4x} \\
\qquad\qquad\qquad\quad 15x^2 + 8x - 12 \\
\qquad\qquad\qquad\quad \underline{15x^2 + 5x - 10} \\
\qquad\qquad\qquad\qquad\qquad 3x - 2)3x^2 + x - 2(x + 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{3x^2 - 2x} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3x - 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{3x - 2}
\end{array}$$

Note.—Each division should be continued until the remainder is of a lower degree than the divisor.

2. Find the H. C. D. of $6x^3 + 21x^2 + 9x$ and $14x^4 + 47x^3 + 7x^2 - 24x$.

By examining these quantities we see that x is a common factor. It may be set aside and inserted after division as a factor of the H. C. D. (Art. 103, Prin.).

We also see that the factor 3 is found in the first quantity, but not in the second. It may therefore be rejected, since it is not a factor of the H. C. D. (Art. 105, Prin. 3).

After removing these factors we have

$$\begin{array}{ll}
 & 2x^2 + 7x + 3)14x^3 + 47x^2 + 7x - 24(7x - 1 \\
 & \qquad\qquad\quad \underline{14x^3 + 49x^2 + 21x} \\
 & \qquad\qquad\qquad\quad - 2x^2 - 14x - 24 \\
 & \qquad\qquad\qquad\quad \underline{- 2x^2 - 7x - 3} \\
\text{(See Art. 105, Prin. 3.)} & \qquad\qquad\qquad\quad - 7)\underline{- 7x - 21} \\
 & \qquad\qquad\qquad\qquad\qquad x + 3)2x^2 + 7x + 3(2x + 1 \\
 & \qquad\qquad\qquad\qquad\qquad\qquad \underline{2x^2 + 6x} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad x + 3 \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{x + 3}
\end{array}$$

Inserting x, the H. C. D. is $x^2 + 3x$.

3. Find the H. C. D. of $20x^3 - 39x^2 - 29x + 48$ and $20x^4 - 9x^3 - 95x^2 + 3x + 81$.

$$\begin{array}{ll}
 & 20x^3 - 39x^2 - 29x + 48)20x^4 - 9x^3 - 95x^2 + 3x + 81(x + 3 \\
 & \qquad\qquad\qquad\qquad\quad \underline{20x^4 - 39x^3 - 29x^2 + 48x} \\
 & \qquad\qquad\qquad\qquad\qquad\quad 30x^3 - 66x^2 - 45x + 81 \\
\text{(See Art. 105, Prin. 3).} & \qquad\qquad\qquad\qquad\qquad\quad \underline{2\qquad\qquad\qquad\qquad\qquad} \\
 & \qquad\qquad\qquad\qquad\qquad\quad 60x^3 - 132x^2 - 90x + 162 \\
 & \qquad\qquad\qquad\qquad\qquad\quad \underline{60x^3 - 117x^2 - 87x + 144} \\
\text{(See Art. 105, Prin. 3.)} & \qquad\qquad\qquad\qquad\qquad - 3)\underline{- 15x^2 - 3x + 18} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad 5x^2 + x - 6
\end{array}$$

$$\begin{array}{l}
5x^2 + x - 6)20x^3 - 39x^2 - 29x + 48(4x - 43 \\
\qquad\qquad\quad \underline{20x^3 + 4x^2 - 24x\qquad\quad} \\
\qquad\qquad\qquad\quad - 43x^2 - 5x + 48 \\
\qquad\qquad\qquad\qquad\ \ \underline{5\qquad\qquad\qquad\quad} \\
\qquad\qquad\qquad\ \ - 215x^2 - 25x + 240 \\
\qquad\qquad\qquad\ \ \underline{- 215x^2 - 43x + 258} \\
\qquad\qquad\qquad\qquad\qquad 18)\underline{18x - 18} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad x - 1)5x^2 + x - 6(5x + 6 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{5x^2 - 5x\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 6x - 6 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{6x - 6}
\end{array}$$

∴ The H. C. D. is $x - 1$.

EXAMPLES.

Find the H. C. D. of—

4. $6x^2 - 19x + 15$ and $14x^2 - 11x - 15$.
5. $5x^2 - 7x - 6$ and $25x^2 - 25x - 24$.
6. $40x^2 + 7x - 66$ and $56x^2 + 61x - 22$.
7. $63x^2 - 122x + 55$ and $72x^2 - 7x - 99$.
8. $3x^3 - 17x^2 + 16x - 4$ and $15x^3 - 7x^2 - 11x + 6$.
9. $3x^3 - 30x^2 + 44x + 8$ and $9x^3 - 20x^2 + 4x$.
10. $12x^4 - 2x^3 - 4x^2 - 30x$ and $24x^3 - 6x^2 - 9x - 54$.
11. $6x^3 - 28x^2 + 24x + 10$ and $5x^3 - 18x^2 + 4x + 3$.
12. $5x^4 + 8x^3 + x^2 - 2x$ and $7x^4 + 11x^3 + x^2 - 3x$.
13. $2x^4 - x^3 - 4x - 3$ and $3x^4 - 4x^3 - x - 2$.
14. $2x^4 - 12x^3 + 17x^2 + 6x - 9$ and $4x^3 - 18x^2 + 19x - 3$.
15. $6x^4 + 25x^3 - 21x^2 + 4x$ and $24x^4 + 112x^3 - 94x^2 + 18x$.
16. $2x^4 - 2x$ and $5x^3 + 12x^2 + 12x + 7$.
17. $x^5 - 17x^3 + 7x^2 - 9$ and $x^5 - 16x^3 + 2x^2 + 5x - 12$.
18. $12x^3 - 88x^2 + 128x$ and $3x^3 - 33x^2 + 96x - 84$.
19. $12x^5 - 51x^3 + 12x$ and $2x^5 - 4x^4 - 2x^3 + 4x^2$.
20. $9a^4 + 12a^3 + 10a^2 + 4a + 1$ and $3a^4 + 8a^3 + 14a^2 + 8a + 3$.

108. To find the H. C. D. of three or more quantities,

RULE.

Find the highest common divisor of any two of them, then of this result and a third quantity, and so on. The last divisor will be the highest common divisor of the given quantities.

EXAMPLES.

Find the H. C. D. of—

1. $10x^2 + 13x - 3$, $8x^2 + 8x - 6$, and $6x^2 + 3x - 9$.

2. $2x^3 + 9x^2 + 7x - 6$, $3x^3 + 11x^2 + 8x + 6$, and $x^3 + 7x^2 + 14x + 6$.

3. $x^3 - 4x^2 + 9x - 10$, $x^3 + 2x^2 - 3x + 20$, and $x^3 + 5x^2 - 9x + 35$.

4. $x^4 - 6x^2 - 7x - 6$, $x^4 - 4x^3 + 2x^2 + x + 6$, and $x^4 + x^3 - 8x^2 - 9x - 9$.

5. $x^3 + 2x^2 + 2x + 1$, $x^3 + 3x^2 + 4x + 2$, and $x^3 + 4x^2 + 6x + 3$.

6. $x^3 - x^2 - 4x + 4$, $x^3 - 3x^2 - x + 6$, and $x^3 - 4x^2 + 3x + 2$.

7. $2x^3 + 5x^2 + 5x + 6$, $3x^3 + 5x^2 - 5x - 6$, and $x^3 + 4x^2 + 5x + 2$.

LOWEST COMMON MULTIPLE.

109. A **Multiple** of a quantity is a quantity that will exactly contain it.

Thus, $3a^2b$ is a multiple of ab.

110. A **Common Multiple** of two or more quantities is a quantity that is a multiple of each of them.

Thus, $3a^2b$ is a common multiple of $3a$ and $3b$.

111. The **Lowest Common Multiple** (L. C. M.) of two or more quantities is a quantity of the lowest degree which is a multiple of each of them.

Thus, $6ab$ is the L. C. M. of $3a$ and $2b$.

PRINCIPLE I.

A multiple of a quantity must contain all the factors of that quantity.

PRINCIPLE II.

A common multiple of two or more quantities must contain all the factors of each quanity.

PRINCIPLE III.

The lowest common multiple of two or more quantities must contain all the factors of each quantity, and no others.

These principles are evident from the definitions given.

CASE I.

112. When the quantities are readily factored.

1. Find the L. C. M. of $6a^2bc^2$ and $9ab^2c$.

PROCESS.

$$\begin{array}{ll}
6a^2bc^2 & = 2 \times 3 \times aa \times b \times cc \\
9ab^2c & = 3 \times 3 \times a \times bb \times c \\
\hline
\text{L. C. M.} & = 9ab^2c \times 2 \times a \times c = 18a^2b^2c^2, \textit{ Ans.}
\end{array}$$

The L. C. M. must contain $9ab^2c$ (ART. 111, Def.) and all the factors of $6a^2bc^2$ not found in $9ab^2c$ (ART. 111, Prin. 3), which are 2, a, and c. Hence, the L. C. M. = $9ab^2c \times 2 \times a \times c = 18a^2b^2c^2$.

2. Find the L. C. M. of $a^4 - b^4$ and $a^6 + b^6$.

PROCESS.

$$\begin{array}{ll}
a^4 - b^4 & = (a^2 + b^2)(a + b)(a - b) \\
a^6 + b^6 & = (a^2 + b^2)(a^4 - a^2b^2 + b^4) \\
\hline
\text{L. C. M.} & = (a^6 + b^6)(a + b)(a - b) = a^8 - a^6b^2 + a^2b^6 - b^8.
\end{array}$$

Therefore, to find the L. C. M. when the quantities are readily factored,

RULE.

Resolve the quantities into their prime factors. Multiply the highest quantity by all the factors of the other quantities not found in the highest quantity.

EXAMPLES.

Find the L. C. M. of—

3. $4x^2y$, $12xy^3$, and $18x^3y^2$.
4. $15x^2yz^2$, $20x^3y^2z$, and $60x^2y^2z^2$.
5. $(3x+1)$, $2(3x-1)$, and $4(9x^2-1)$.
6. x^2-y^2, x^3-y^3, and x^6-y^6.
7. x^3-y^3, $2(x^2y-xy^2)^2$, and x^6-y^6.
8. x^2-y^2, x^4+y^4, and x^8-y^8.
9. x^2-5x+6 and x^2-6x+8.
10. x^2-9, x^2+x-12, and $x^2+2x-15$.
11. $x+1$, x^2-1, and x^3-1.
12. x^2+y^2, x^6+y^6, x^6-y^6, and $x^{12}-y^{12}$.
13. $2x^2-7x-4$, $4x^2+10x+4$, and $6x^2-7x-5$.
14. $(x+y)^2$, $(x-y)^2$, and x^2-y^2.
15. a^2+ab+b^2, a^2-ab+b^2, and $a^4+a^2b^2+b^4$.
16. $(a+b)^2-c^2$, $(a+c)^2-b^2$, $a^2-(b+c)^2$.
17. $x^{12}+y^{12}$, x^6+y^6, and x^8-y^8.
18. $10x^2-41x+21$ and $6x^2-31x+35$.
19. $7x^2+11xy-6y^2$ and $5x^2+15xy+10y^2$.
20. $2x^2-5xy+2y^2$ and $6x^2-4x-3xy+2y$.
21. $4x^2-xy-3y^2$ and $3x^3-3x^2y+xy^2-y^3$.
22. x^2+xy+y^2, x^3-y^3, and $x-y$.
23. $1-x+x^2$, $1+x+x^2$, and $1+x^2+x^4$.
24. $x^2+2xy+y^2$, $x^2-2xy+y^2$, and $x^4-2x^2y^2+y^4$.
25. x^2+x, x^4-x^2, and x^6+x^3.
26. x^2-y^2, x^4-y^4, and x^6+y^6.
27. x^2-1, x^3+1, x^3-1, and x^6+1.
28. $x^3+x^2y+xy^2+y^3$ and $x^3-x^2y+xy^2-y^3$.

CASE II.

113. When the quantities are not readily factored.

PRINCIPLE.

The lowest common multiple of two quantities equals either quantity divided by their highest common divisor, and this quotient multiplied by the other quantity.

Let A and B be any two quantities, c their highest common divisor, and m and n the other two factors, respectively; then

$$A = m \times c \qquad (1)$$

$$\underline{B = n \times c} \qquad (2)$$

And L. C. M. $= B \times m$ (Case I.).

But from (1), $m = \frac{A}{c}$

$\therefore$ L. C. M. $= B \times \frac{A}{c}$.

1. Find the L. C. M. of $x^3 - 3x^2 + 3x - 2$ and $2x^3 - 3x^2 - 3x + 2$.

We first find the H. C. D.

$$x^3 - 3x^2 + 3x - 2)2x^3 - 3x^2 - 3x + 2(2$$

$$\underline{2x^3 - 6x^2 + 6x - 4}$$

$$3)\underline{3x^2 - 9x + 6}$$

$$x^2 - 3x + 2)x^3 - 3x^2 + 3x - 2(x$$

$$\underline{x^3 - 3x^2 + 2x}$$

$$x - 2$$

$$x - 2)x^2 - 3x + 2(x - 1$$

$$\underline{x^2 - 2x}$$

$$-\ x + 2$$

$$\underline{-\ x + 2}$$

H. C. D. $= x - 2$.

Then, by Art. 113, Prin.—

$$\text{the L. C. M.} = x^3 - 3x^2 + 3x - 2 \times \frac{2x^3 - 3x^2 - 3x + 2}{x - 2}$$

$$= 2x^5 - 5x^4 + 2x^3 + 2x^2 - 5x + 2, \textit{ Ans.}$$

EXAMPLES.

Find the factors of the L. C. M. of—

2. $4x^3 + x^2 + 12x + 3$ and $x^3 + 2x^2 + 3x + 6$.

3. $2x^2 - 5xy + 3y^2$ and $2x^3 - 3x^2y - 2xy^2 + 3y^3$.

4. $x^3 - 6x^2 + 11x - 6$ and $x^3 - 9x^2 + 26x - 24$.

5. $x^3 - 3x^2 + 3x - 2$ and $2x^3 - 3x^2 - 3x + 2$.

6. $x^3 - 6x^2 + 11x - 6$ and $x^3 - 19x + 30$.

7. $8x^3 - 14x^2 - 19x + 30$ and $12x^3 + 8x^2 - 43x - 42$.

8. $20x^3 - 8x^2 - 125x + 50$ and $12x^3 + 8x^2 - 75x - 50$.

9. $2x^3 + 7x^2 - 10x - 24$ and $4x^3 + 12x^2 - 31x - 60$.

10. $3x^4 + x^3 - 5x^2 - 2x - 2$ and $12x^4 + 4x^3 - 5x^2 - 3x - 3$.

11. $x^5 + 2x^4 + 3x^3 + 3x^2 + 2x + 1$ and $x^5 - x^3 - x^2 - 2x - 1$.

12. $m^4 + m^3 + 2m^2 + m + 1$ and $m^4 - 1$.

13. $3m^4 - m^3 - 3m + 1$ and $5m^4 - 2m^3 - 5m + 2$.

14. $n^3 - 2n^2 - 5n + 6$ and $n^3 - 5n^2 + 8n - 6$.

15. $6x^4 - 5x^2 + 5x - 1$ and $8x^4 - 2x^3 + x + 1$.

16. $m^3 - 7m - 6$ and $m^3 + m^2 - 9m - 9$.

114. To find the L. C. M. of three or more quantities.

RULE.

Find the lowest common multiple of any two of them, then of this result and a third quantity, and so on.

EXAMPLES.

Find the factors of the L. C. M. of—

1. $x - y$, $x + y$, $x^2 - y^2$, $x^3 - y^3$, and $x^4 - y^4$.

2. $6x^2 + 4x - 16$, $9x^2 - 27x + 20$, and $12x^2 + 5x - 28$.

3. $5x^2 - 14x + 8$, $3x^2 - 4x - 4$, and $x^2 - 4x + 4$.

4. $x^4 + 5x^3 + 5x^2 - 5x - 6$, $x^3 + 6x^2 + 11x + 6$, and $x^3 + 4x^2 + x - 6$.

5. $x^4 - 2x^3 - x^2 + 2x$, $x^4 - 4x^3 + 5x^2 - 2x$, and $x^3 - x^2 - 4x + 4$.

6. $n^3 - 7n + 6$, $n^3 + 3n^2 - 16n + 12$, and $n^3 - 4n^2 + 5n - 2$.

V. FRACTIONS.

115. A **Fraction** is an indicated expression of division.

An *Algebraic Fraction* is written as in Arithmetic. The dividend, called the *Numerator*, is written above, and the divisor, called the *Denominator*, is written below a horizontal line.

Thus, $\frac{3a - 2b}{c + 4d}$ means $(3a - 2b) \div (c + 4d)$.

116. The **Terms of a Fraction** are the numerator and the denominator.

117. An **Integral Quantity** is one that does not have the fractional form; as, $m + n$.

118. A **Mixed Quantity** is one that is partly integral and partly fractional; as, $a + \frac{b}{c}$, or $a - \frac{b}{c}$.

PRINCIPLES OF FRACTIONS.

PRINCIPLE I.

119. *Multiplying the numerator multiplies the value of the fraction.*

Let $\frac{n}{d}$ be any fraction, q its value, and x any quantity; then

$$\frac{n}{d} = q.$$

Since the dividend equals the product of the divisor and quotient,

$$n = qd.$$

And $nx = qxd$ (Art. 69).

Hence $\frac{nx}{d} = qx$ (Art. 69).

PRINCIPLE II.

Dividing the numerator divides the value of the fraction.

Let $\frac{n}{d} = q.$

Then $n = qd.$

And $n \div x = (q \div x)d$ (Art. 69).

Hence $\frac{n \div x}{d} = q \div x$ (Art. 69).

PRINCIPLE III.

Multiplying the denominator divides the value of the fraction.

Let $\frac{n}{d} = q.$

Then $n = qd.$

And $n = (q \div x)dx$ (Art. 54, Ax. 6).

Hence $\frac{n}{dx} = q \div x$ (Art. 69).

PRINCIPLE IV.

Dividing the denominator multiplies the value of the fraction.

Let $\frac{n}{d} = q.$

Then $n = qd.$

And $n = qx(d \div x)$ (Art. 54, Ax. 6).

Hence $\frac{n}{d \div x} = qx$ (Art. 69).

PRINCIPLE V.

Multiplying both numerator and denominator by the same quantity does not change the value of the fraction.

Let $\frac{n}{d} = q.$

Then $n = qd.$

And $nx = qdx$ (Art. 69).

Hence $\frac{nx}{dx} = q$ (Art. 69).

PRINCIPLE VI.

Dividing both numerator and denominator by the same quantity does not change the value of the fraction.

Let $\frac{n}{d} = q.$

Then $n = qd.$

And $n \div x = q(d \div x)$ (ART. 69).

Hence $\frac{n \div x}{d \div x} = q$ (ART. 69).

PRINCIPLE VII.

Changing the sign of both numerator and denominator does not change the value of the fraction.

This is the same as multiplying both numerator and denominator by -1, which does not change the value of a fraction (Prin. 5).

PRINCIPLE VIII.

Changing the sign of either numerator or denominator changes the sign of the fraction.

This is the same as multiplying or dividing a fraction by -1, which will change the sign of its value (Prin. 1 and 2).

Thus, $\frac{a}{b} = -\frac{-a}{b} = -\frac{a}{-b}.$

REDUCTION OF FRACTIONS.

CASE I.

120. To reduce a fraction to its lowest terms.

A fraction is in its *lowest terms* when its numerator and denominator are prime to each other.

Since dividing both numerator and denominator by the same quantity does not change the value of the fraction (ART. 119, Prin. 6), we have the following

RULE.

Resolve both terms of the fraction into their prime factors, and cancel all factors that are common to both terms. Or,

Divide both terms by their highest common divisor.

1. Reduce $\frac{x^2-y^2}{(x+y)^2}$ to its lowest terms.

$$\frac{x^2-y^2}{(x+y)^2}=\frac{(x+y)(x-y)}{(x+y)(x+y)}=\frac{x-y}{x+y},\ \textit{Ans.}$$

2. Reduce $\frac{x^2+7x+12}{x^2+9x+20}$ to its lowest terms.

$$\frac{x^2+7x+12}{x^2+9x+20}=\frac{(x+4)(x+3)}{(x+4)(x+5)}=\frac{x+3}{x+5},\ \textit{Ans.}$$

EXAMPLES.

Reduce the following to their lowest terms:

3. $\frac{8a^4}{12a^3}$.

4. $\frac{24m^4n^7}{36m^2n^8}$.

5. $\frac{39a^7b^8c^6}{65a^6b^3c^7}$.

6. $\frac{132x^2y^3z^6}{144x^3yz^7}$.

7. $\frac{121m^3n^4w^2}{154m^4n^2w^2}$.

8. $\frac{52a^4m^3n^7x^9}{169a^8m^3n^{10}x^9}$.

9. $\frac{90m^8n^{12}x^9y^{11}z^4}{135m^2n^{12}x^9y^2z^6}$.

10. $\frac{4a^2+8a}{4a^2-8a}$.

11. $\frac{3x^2+6x}{6x^2-9x}$.

12. $\frac{5a^3+5a^2+5a}{5a^3-5a^2-5a}$.

13. $\frac{3a^3x+6a^2bx+3ab^2x}{3ax^3-6ax^2y+3axy^2}$.

14. $\frac{x^2-y^2}{x^2-2xy+y^2}$.

15. $\frac{a^2-b^2}{a^2+2ab+b^2}$.

16. $\frac{a^4-b^4}{a^4+2a^2b^2+b^4}$.

17. $\frac{a^6+b^9}{a^4-b^6}$.

18. $\frac{a^3+b^3}{a^6-b^6}$.

19. $\frac{(x^2-a^2)^2}{(x+a)^4}$.

20. $\frac{(m^2-n^2)^2}{(m+n)^3}$.

21. $\frac{a^3+b^3}{(a+b)^3}$.

22. $\frac{a^2-3a-28}{a^2+10a+24}$.

23. $\dfrac{m^2+5m-36}{m^2-13m+36}$.

24. $\dfrac{a^2-17a+72}{a^2-18a+80}$.

25. $\dfrac{x^2+17x+66}{x^2+18x+72}$.

26. $\dfrac{6x^2+x-12}{10x^2+23x+12}$.

27. $\dfrac{40x^2-15x}{16x^2+34x-15}$.

28. $\dfrac{m^4-m^2n^2+n^4}{m^6+n^6}$.

29. $\dfrac{x^2-(y-z)^2}{(x+y)^2-z^2}$.

30. $\dfrac{(m+n)^2-a^2}{m^2-(n+a)^2}$.

31. Reduce $\dfrac{3x^3-17x^2+16x-4}{15x^3-7x^2-11x+6}$ to its lowest terms.

By Art. 107 the H. C. D. of $3x^3-17x^2+16x-4$ and $15x^3-7x^2-11x+6$ is $3x-2$. Dividing the numerator by $3x-2$, the quotient is x^2-5x+2; and dividing the denominator by $3x-2$, the quotient is $5x^2+x-3$. Hence

$$\frac{3x^3-17x^2+16x-4}{15x^3-7x^2-11x+6}=\frac{x^2-5x+2}{5x^2+x-3}, \textit{ Ans.}$$

Reduce by finding the H. C. D. of the terms:

32. $\dfrac{3x^3-14x^2+12x+8}{9x^3-20x^2+4x}$.

33. $\dfrac{6x^3-28x^2+24x+10}{5x^3-18x^2+4x+3}$.

34. $\dfrac{2x^4-x^3-4x-3}{3x^4-4x^3-x-2}$.

35. $\dfrac{6x^3+5x^2-4x-3}{8x^3+8x^2-2x-2}$.

36. $\dfrac{10m^3-7m^2-6m-9}{6m^3-17m^2+18m-9}$.

37. $\dfrac{m^3-2m^2-5m-12}{m^3-3m^2-7m-15}$.

CASE II.

121. To reduce a fraction to an integral or mixed quantity.

Since a fraction is an indicated expression of division (Art. 115), we have the following

RULE.

Divide the numerator by the denominator.

EXAMPLES.

Find the L. C. M. of—

3. $4x^2y$, $12xy^3$, and $18x^3y^2$.
4. $15x^2yz^2$, $20x^3y^2z$, and $60x^2y^2z^2$.
5. $(3x+1)$, $2(3x-1)$, and $4(9x^2-1)$.
6. x^2-y^2, x^3-y^3, and x^6-y^6.
7. x^3-y^3, $2(x^2y-xy^2)^2$, and x^6-y^6.
8. x^2-y^2, x^4+y^4, and x^8-y^8.
9. x^2-5x+6 and x^2-6x+8.
10. x^2-9, x^2+x-12, and $x^2+2x-15$.
11. $x+1$, x^2-1, and x^3-1.
12. x^2+y^2, x^6+y^6, x^6-y^6, and $x^{12}-y^{12}$.
13. $2x^2-7x-4$, $4x^2+10x+4$, and $6x^2-7x-5$.
14. $(x+y)^2$, $(x-y)^2$, and x^2-y^2.
15. a^2+ab+b^2, a^2-ab+b^2, and $a^4+a^2b^2+b^4$.
16. $(a+b)^2-c^2$, $(a+c)^2-b^2$, $a^2-(b+c)^2$.
17. $x^{12}+y^{12}$, x^6+y^6, and x^8-y^8.
18. $10x^2-41x+21$ and $6x^2-31x+35$.
19. $7x^2+11xy-6y^2$ and $5x^2+15xy+10y^2$.
20. $2x^2-5xy+2y^2$ and $6x^2-4x-3xy+2y$.
21. $4x^2-xy-3y^2$ and $3x^3-3x^2y+xy^2-y^3$.
22. x^2+xy+y^2, x^3-y^3, and $x-y$.
23. $1-x+x^2$, $1+x+x^2$, and $1+x^2+x^4$.
24. $x^2+2xy+y^2$, $x^2-2xy+y^2$, and $x^4-2x^2y^2+y^4$.
25. x^2+x, x^4-x^2, and x^6+x^3.
26. x^2-y^2, x^4-y^4, and x^6+y^6.
27. x^2-1, x^3+1, x^3-1, and x^6+1.
28. $x^3+x^2y+xy^2+y^3$ and $x^3-x^2y+xy^2-y^3$.

CASE II.

113. When the quantities are not readily factored.

PRINCIPLE.

The lowest common multiple of two quantities equals either quantity divided by their highest common divisor, and this quotient multiplied by the other quantity.

Let A and B be any two quantities, c their highest common divisor, and m and n the other two factors, respectively; then

$$A = m \times c \qquad (1)$$

$$\underline{B = n \times c} \qquad (2)$$

$$\text{And L. C. M.} = B \times m \text{ (Case I.).}$$

$$\text{But from (1), } m = \frac{A}{c}$$

$$\therefore \text{L. C. M.} = B \times \frac{A}{c}.$$

1. Find the L. C. M. of $x^3 - 3x^2 + 3x - 2$ and $2x^3 - 3x^2 - 3x + 2$.

We first find the H. C. D.

$$x^3 - 3x^2 + 3x - 2)2x^3 - 3x^2 - 3x + 2(2$$

$$\underline{2x^3 - 6x^2 + 6x - 4}$$

$$3)\underline{3x^2 - 9x + 6}$$

$$x^2 - 3x + 2)x^3 - 3x^2 + 3x - 2(x$$

$$\underline{x^3 - 3x^2 + 2x}$$

$$x - 2$$

$$x - 2)x^2 - 3x + 2(x - 1$$

$$\underline{x^2 - 2x}$$

$$- x + 2$$

$$\text{H. C. D.} = x - 2. \qquad \underline{- x + 2}$$

Then, by Art. 113, Prin.—

$$\text{the L. C. M.} = x^3 - 3x^2 + 3x - 2 \times \frac{2x^3 - 3x^2 - 3x + 2}{x - 2}$$

$$= 2x^5 - 5x^4 + 2x^3 + 2x^2 - 5x + 2, \textit{ Ans.}$$

EXAMPLES.

Find the factors of the L. C. M. of—

2. $4x^3 + x^2 + 12x + 3$ and $x^3 + 2x^2 + 3x + 6$.

3. $2x^2 - 5xy + 3y^2$ and $2x^3 - 3x^2y - 2xy^2 + 3y^3$.

4. $x^3 - 6x^2 + 11x - 6$ and $x^3 - 9x^2 + 26x - 24$.

5. $x^3 - 3x^2 + 3x - 2$ and $2x^3 - 3x^2 - 3x + 2$.

6. $x^3 - 6x^2 + 11x - 6$ and $x^3 - 19x + 30$.

7. $8x^3 - 14x^2 - 19x + 30$ and $12x^3 + 8x^2 - 43x - 42$.

8. $20x^3 - 8x^2 - 125x + 50$ and $12x^3 + 8x^2 - 75x - 50$.

9. $2x^3 + 7x^2 - 10x - 24$ and $4x^3 + 12x^2 - 31x - 60$.

10. $3x^4 + x^3 - 5x^2 - 2x - 2$ and $12x^4 + 4x^3 - 5x^2 - 3x - 3$.

11. $x^5 + 2x^4 + 3x^3 + 3x^2 + 2x + 1$ and $x^5 - x^3 - x^2 - 2x - 1$.

12. $m^4 + m^3 + 2m^2 + m + 1$ and $m^4 - 1$.

13. $3m^4 - m^3 - 3m + 1$ and $5m^4 - 2m^3 - 5m + 2$.

14. $n^3 - 2n^2 - 5n + 6$ and $n^3 - 5n^2 + 8n - 6$.

15. $6x^4 - 5x^3 + 5x - 1$ and $8x^4 - 2x^3 + x + 1$.

16. $m^3 - 7m - 6$ and $m^3 + m^2 - 9m - 9$.

114. To find the L. C. M. of three or more quantities.

RULE.

Find the lowest common multiple of any two of them, then of this result and a third quantity, and so on.

EXAMPLES.

Find the factors of the L. C. M. of—

1. $x - y$, $x + y$, $x^2 - y^2$, $x^3 - y^3$, and $x^4 - y^4$.

2. $6x^2 + 4x - 16$, $9x^2 - 27x + 20$, and $12x^2 + 5x - 28$.

3. $5x^2 - 14x + 8$, $3x^2 - 4x - 4$, and $x^2 - 4x + 4$.

4. $x^4 + 5x^3 + 5x^2 - 5x - 6$, $x^3 + 6x^2 + 11x + 6$, and $x^3 + 4x^2 + x - 6$.

5. $x^4 - 2x^3 - x^2 + 2x$, $x^4 - 4x^3 + 5x^2 - 2x$, and $x^3 - x^2 - 4x + 4$.

6. $n^3 - 7n + 6$, $n^3 + 3n^2 - 16n + 12$, and $n^3 - 4n^2 + 5n - 2$.

V. FRACTIONS.

115. A **Fraction** is an indicated expression of division.

An *Algebraic Fraction* is written as in Arithmetic. The dividend, called the *Numerator*, is written above, and the divisor, called the *Denominator*, is written below a horizontal line.

Thus, $\dfrac{3a-2b}{c+4d}$ means $(3a-2b) \div (c+4d)$.

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PRINCIPLES OF FRACTIONS.

PRINCIPLE I.

119. *Multiplying the numerator multiplies the value of the fraction.*

Let $\dfrac{n}{d}$ be any fraction, q its value, and x any quantity; then

$$\frac{n}{d}=q.$$

Since the dividend equals the product of the divisor and quotient,

$$n=qd.$$

And $$nx=qxd \text{ (Art. 69).}$$

Hence $$\frac{nx}{d}=qx \text{ (Art. 69).}$$

PRINCIPLE II.

Dividing the numerator divides the value of the fraction.

Let $\frac{n}{d} = q$.

Then $n = qd$.

And $n \div x = (q \div x)d$ (Art. 69).

Hence $\frac{n \div x}{d} = q \div x$ (Art. 69).

PRINCIPLE III.

Multiplying the denominator divides the value of the fraction.

Let $\frac{n}{d} = q$.

Then $n = qd$.

And $n = (q \div x)dx$ (Art. 54, Ax. 6).

Hence $\frac{n}{dx} = q \div x$ (Art. 69).

PRINCIPLE IV.

Dividing the denominator multiplies the value of the fraction.

Let $\frac{n}{d} = q$.

Then $n = qd$.

And $n = qx(d \div x)$ (Art. 54, Ax. 6).

Hence $\frac{n}{d \div x} = qx$ (Art. 69).

PRINCIPLE V.

Multiplying both numerator and denominator by the same quantity does not change the value of the fraction.

Let $\frac{n}{d} = q$.

Then $n = qd$.

And $nx = qdx$ (Art. 69).

Hence $\frac{nx}{dx} = q$ (Art. 69).

PRINCIPLE VI.

Dividing both numerator and denominator by the same quantity does not change the value of the fraction.

Let $\frac{n}{d} = q.$

Then $n = qd.$

And $n \div x = q(d \div x)$ (ART. 69).

Hence $\frac{n \div x}{d \div x} = q$ (ART. 69).

PRINCIPLE VII.

Changing the sign of both numerator and denominator does not change the value of the fraction.

This is the same as multiplying both numerator and denominator by -1, which does not change the value of a fraction (Prin. 5).

PRINCIPLE VIII.

Changing the sign of either numerator or denominator changes the sign of the fraction.

This is the same as multiplying or dividing a fraction by -1, which will change the sign of its value (Prin. 1 and 2).

Thus, $\frac{a}{b} = -\frac{-a}{b} = -\frac{a}{-b}.$

REDUCTION OF FRACTIONS.

CASE I.

120. To reduce a fraction to its lowest terms.

A fraction is in its *lowest terms* when its numerator and denominator are prime to each other.

Since dividing both numerator and denominator by the same quantity does not change the value of the fraction (ART. 119, Prin. 6), we have the following

RULE.

Resolve both terms of the fraction into their prime factors, and cancel all factors that are common to both terms. Or,

Divide both terms by their highest common divisor.

1. Reduce $\frac{x^2 - y^2}{(x+y)^2}$ to its lowest terms.

$$\frac{x^2 - y^2}{(x+y)^2} = \frac{(x+y)(x-y)}{(x+y)(x+y)} = \frac{x-y}{x+y}, \text{ Ans.}$$

2. Reduce $\frac{x^2 + 7x + 12}{x^2 + 9x + 20}$ to its lowest terms.

$$\frac{x^2 + 7x + 12}{x^2 + 9x + 20} = \frac{(x+4)(x+3)}{(x+4)(x+5)} = \frac{x+3}{x+5}, \text{ Ans.}$$

EXAMPLES.

Reduce the following to their lowest terms:

3. $\frac{8a^4}{12a^3}$.

4. $\frac{24m^4n^7}{36m^2n^8}$.

5. $\frac{39a^7b^3c^6}{65a^6b^3c^7}$.

6. $\frac{132x^2y^3z^6}{144x^2yz^7}$.

7. $\frac{121m^3n^4w^2}{154m^4n^2w^2}$.

8. $\frac{52a^4m^3n^7x^9}{169a^8m^3n^{10}x^9}$.

9. $\frac{90m^8n^{12}x^9y^{11}z^4}{135m^2n^{12}x^9y^2z^6}$.

10. $\frac{4a^2 + 8a}{4a^2 - 8a}$.

11. $\frac{3x^2 + 6x}{6x^2 - 9x}$.

12. $\frac{5a^3 + 5a^2 + 5a}{5a^3 - 5a^2 - 5a}$.

13. $\frac{3a^3x + 6a^2bx + 3ab^2x}{3ax^3 - 6ax^2y + 3axy^2}$.

14. $\frac{x^2 - y^2}{x^2 - 2xy + y^2}$.

15. $\frac{a^2 - b^2}{a^2 + 2ab + b^2}$.

16. $\frac{a^4 - b^4}{a^4 + 2a^2b^2 + b^4}$.

17. $\frac{a^6 + b^9}{a^4 - b^6}$.

18. $\frac{a^3 + b^3}{a^6 - b^6}$.

19. $\frac{(x^2 - a^2)^2}{(x+a)^4}$.

20. $\frac{(m^2 - n^2)^2}{(m+n)^3}$.

21. $\frac{a^3 + b^3}{(a+b)^3}$.

22. $\frac{a^2 - 3a - 28}{a^2 + 10a + 24}$.

23. $\dfrac{m^2+5m-36}{m^2-13m+36}$.

24. $\dfrac{a^2-17a+72}{a^2-18a+80}$.

25. $\dfrac{x^2+17x+66}{x^2+18x+72}$.

26. $\dfrac{6x^2+x-12}{10x^2+23x+12}$.

27. $\dfrac{40x^2-15x}{16x^2+34x-15}$.

28. $\dfrac{m^4-m^2n^2+n^4}{m^6+n^6}$.

29. $\dfrac{x^2-(y-z)^2}{(x+y)^2-z^2}$.

30. $\dfrac{(m+n)^2-a^2}{m^2-(n+a)^2}$.

31. Reduce $\dfrac{3x^3-17x^2+16x-4}{15x^3-7x^2-11x+6}$ to its lowest terms.

By Art. 107 the H. C. D. of $3x^3-17x^2+16x-4$ and $15x^3-7x^2-11x+6$ is $3x-2$. Dividing the numerator by $3x-2$, the quotient is x^2-5x+2; and dividing the denominator by $3x-2$, the quotient is $5x^2+x-3$. Hence

$$\frac{3x^3-17x^2+16x-4}{15x^3-7x^2-11x+6}=\frac{x^2-5x+2}{5x^2+x-3},\ \textit{Ans.}$$

Reduce by finding the H. C. D. of the terms:

32. $\dfrac{3x^3-14x^2+12x+8}{9x^3-20x^2+4x}$.

33. $\dfrac{6x^3-28x^2+24x+10}{5x^3-18x^2+4x+3}$.

34. $\dfrac{2x^4-x^3-4x-3}{3x^4-4x^3-x-2}$.

35. $\dfrac{6x^3+5x^2-4x-3}{8x^3+8x^2-2x-2}$.

36. $\dfrac{10m^3-7m^2-6m-9}{6m^3-17m^2+18m-9}$.

37. $\dfrac{m^3-2m^2-5m-12}{m^3-3m^2-7m-15}$.

CASE II.

121. To reduce a fraction to an integral or mixed quantity.

Since a fraction is an indicated expression of division (Art. 115), we have the following

RULE.

Divide the numerator by the denominator.

1. Reduce $\frac{a^3+b^3}{a+b}$ to an integral quantity.

$$\frac{a^3+b^3}{a+b} = a^2 - ab + b^2 \text{ (Art. 83), } \textit{Ans.}$$

2. Reduce $\frac{x^3-x-12}{x-4}$ to a mixed quantity.

$$\begin{array}{l} x-4)x^3 - x - 12(x^2+4x+15+\dfrac{48}{x-4}, \textit{ Ans.} \\ \quad\underline{x^3-4x^2} \\ \qquad 4x^2 - x \\ \qquad \underline{4x^2 - 16x} \\ \qquad\qquad 15x - 12 \\ \qquad\qquad \underline{15x - 60} \\ \qquad\qquad\qquad 48 \end{array}$$

EXAMPLES.

Reduce to integral or mixed quantities—

3. $\frac{x^3+y^3}{x-y}$.

4. $\frac{x^5+y^3}{x+y}$.

5. $\frac{m^8-n^8}{m^3-n^3}$.

6. $\frac{m^2+mn+n^2}{m+n}$.

7. $\frac{x^2+xy+y^2}{x-y}$.

8. $\frac{a^{12}+b^{12}}{a^4+b^4}$.

9. $\frac{a^{3n}+b^{6n}}{a^n+b^{2n}}$.

10. $\frac{a^3+b^3}{(a+b)^3}$.

11. $\frac{a^3+b^3-c^3}{a-c}$.

12. $\frac{na^2-nb^2}{a-b}$.

13. $\frac{a^6+b^6}{a^2-b^2}$.

14. $\frac{a^3}{a-b}$.

15. $\frac{x^4+y^4}{x+y}$.

16. $\frac{x^2-a^2}{x-y}$.

17. $\frac{x^3+4x+5}{x+3}$.

18. $\frac{5x^2+6x+7}{x+4}$.

19. $\dfrac{12x^3 + 8x^2 - 9}{4x^2 + 1}$.

20. $\dfrac{15m^3 + 10m^2 - 40}{5m^2 - 4}$.

21. $\dfrac{4a^3 + 3a^2 + 4a + 5}{a^2 + a + 1}$.

22. $\dfrac{21x^4 + 20x^3 + 30x - 50}{3x^2 + 5x + 4}$.

23. $\dfrac{x^4 + 2x^3 - x^2 - 4x - 6}{x^2 - 2}$.

24. $\dfrac{x^{11} + y^{11}}{x + y}$.

CASE III.

122. To reduce a mixed quantity to a fraction.

This process being the converse of ART. 121, we have the following

RULE.

Multiply the integral part by the denominator of the fraction; to this product add the numerator when the sign before the fraction is +, and subtract it when the sign is −; and write the result over the denominator.

1. Reduce $x + y - \dfrac{a^2 - y^2}{x - y}$ to a fraction.

$$x + y - \frac{a^2 - y^2}{x - y} = \frac{(x + y)(x - y) - (a^2 - y^2)}{x - y}$$
$$= \frac{x^2 - y^2 - a^2 + y^2}{x - y}$$
$$= \frac{x^2 - a^2}{x - y}, \textit{ Ans.}$$

EXAMPLES.

Reduce the following to fractional forms:

2. $1 + \dfrac{x + y}{x - y}$.

3. $x - 1 - \dfrac{x - 1}{x + 1}$.

4. $a + b - \dfrac{a^2 + b^2}{a + b}$.

5. $1 + \dfrac{x^2 + y^2 - z^2}{2xy}$.

6. $x - 1 - \dfrac{x - 1}{x}$.

7. $a + b - \dfrac{(a - b)^2}{a + b}$.

8. $x - y - \dfrac{x^2 + y^2}{x - y}$.

9. $m + n - \dfrac{2mn}{m + n}$.

10. $\dfrac{x^2 + 16}{x - 4} - x - 4$.

11. $\dfrac{2mn + n^2}{m + n} + m - n$.

12. $a - 8 - \dfrac{b^2 - 64}{a + 8}$.

13. $4x + 1 - \dfrac{3x^2}{2x - 1}$.

14. $a^2 + 2ab + b^2 - \dfrac{a^3 - 3a^2b + 3ab^2 - b^3}{a + b}$.

15. $x^2 + 3x - 5 - \dfrac{x^3 + 8x^2 + 10x - 30}{x + 5}$.

16. $x^2 + xy + y^2 - \dfrac{x^4 + y^4}{x^2 - xy + y^2}$.

17. $2x - 3y + 6 + \dfrac{xy + 8x + 3y^2 + 6}{x + y - 1}$.

18. $x^2 - 4x + 10 - \dfrac{x^4 - 3x^3 + 4x^2 + 13x - 9}{x^2 + x - 1}$.

CASE IV.

123. To reduce fractions to their lowest common denominator.

1. Reduce $\dfrac{x^2 - 1}{x^2 + 1}$ and $\dfrac{x^2 + 1}{x^2 - 1}$ to their lowest common denominator.

It is evident that the lowest common denominator is the lowest common multiple of the denominators $x^2 + 1$ and $x^2 - 1$, which is $x^4 - 1$ (Art. 112).

Since both terms of a fraction may be multiplied by the same quantity without changing its value (Art. 119, Prin. 5), we multiply both terms of the first fraction by $(x^4 - 1) \div (x^2 + 1)$, which is $x^2 - 1$; and both terms of the second fraction by $(x^4 - 1) \div (x^2 - 1)$, which is $x^2 + 1$. Hence

$$\frac{x^2 - 1}{x^2 + 1} = \frac{(x^2 - 1)(x^2 - 1)}{(x^2 + 1)(x^2 - 1)} = \frac{(x^2 - 1)^2}{x^4 - 1}.$$

And $$\frac{x^2 + 1}{x^2 - 1} = \frac{(x^2 + 1)(x^2 + 1)}{(x^2 - 1)(x^2 + 1)} = \frac{(x^2 + 1)^2}{x^4 - 1}.$$

Therefore, the required fractions are

$$\frac{(x^2 - 1)^2}{x^4 - 1} \text{ and } \frac{(x^2 + 1)^2}{x^4 - 1}, \textit{ Ans.}$$

Hence to reduce fractions to their lowest common denominator, we have the following

RULE.

Find the lowest common multiple of the denominators, divide this by the denominator of each fraction, and multiply the numerator by the quotient.

NOTE.—Reduce each fraction to its lowest terms before beginning the process.

EXAMPLES.

Reduce the following to equivalent fractions having the lowest common denominator:

2. $\dfrac{a}{mn}$, $\dfrac{b}{m^2}$, and $\dfrac{c}{mn^2}$.

3. $\dfrac{c}{4a}$, $\dfrac{6}{24c}$, and $\dfrac{2n}{3ac}$.

4. $\dfrac{8a}{2a^2c}$, $\dfrac{5c}{bc^2}$, and $\dfrac{12ab}{2a^2b^2}$.

5. $\dfrac{a}{a+b}$, $\dfrac{b}{a-b}$, and $\dfrac{c}{a^2-b^2}$.

6. $\dfrac{a-b}{a+b}$, $\dfrac{a+b}{a-b}$, and $\dfrac{a^2-b^2}{a^2+b^2}$.

7. $\dfrac{x}{(a^2-b^2)}$, $\dfrac{y}{(a+b)^2}$, and $\dfrac{z}{(a-b)^2}$.

8. $\dfrac{2x}{x^2-xy+y^2}$, $\dfrac{3y}{x^3+y^3}$, and $\dfrac{4z}{x+y}$.

9. $\dfrac{1}{m-n}$, $\dfrac{2m(m+n)}{m^2-n^2}$, and $\dfrac{m+n}{m^3-n^3}$.

10. $\dfrac{l}{(a-b)(b-c)}$, $\dfrac{m}{(b-a)(a-c)}$, and $\dfrac{n}{(c-b)(c-a)}$.

11. $\dfrac{a}{(x-a)(y-b)}$, $\dfrac{b}{(a-x)(y-b)}$, and $\dfrac{c}{(x-a)(b-y)}$.

12. $\frac{x-1}{x+2}$, $\frac{2x+3}{2x+4}$, and $\frac{3x-4}{3x-6}$.

13. $\frac{1}{x^2+x-6}$, $\frac{2}{x^2+5x+6}$, and $\frac{3}{x^2-4}$.

14. $\frac{1}{(x-1)(x-2)}$, $\frac{2}{(1-x)(x-3)}$, and $\frac{3}{(2-x)(3-x)}$.

15. $\frac{8}{x+2}$, $\frac{10}{x^2+4x+4}$, and $\frac{12}{x^3+6x^2+12x+8}$.

16. $\frac{a}{x^2+xy+y^2}$, $\frac{b}{x^2-xy+y^2}$, and $\frac{c}{x^4+x^2y^2+y^4}$.

17. $\frac{m}{1-x}$, $\frac{n}{(1-x)^2}$, and $\frac{l}{(1-x)^3}$.

CLEARING EQUATIONS OF FRACTIONS.

124. To clear an equation of fractions.

1. Solve the equation $\frac{2x}{5}-\frac{3}{10}=\frac{x}{15}+\frac{11}{30}$.

SOLUTION.

Since both members of an equation may be multiplied by the same number without changing the equality (ART. 69), we multiply both members of

$$\frac{2x}{5}-\frac{3}{10}=\frac{x}{15}+\frac{11}{30}$$

by the L. C. M. of the denominators, 5, 10, 15, and 30, which is 30, and we have

$$12x-9=2x+11.$$

Whence $\qquad 10x=20$

$$x=2,\ \textit{Ans.}$$

Therefore, to clear an equation of fractions,

RULE.

Multiply each term of the equation by the lowest common multiple of the denominators.

EXAMPLES.

Solve the following equations:

2. $x+\frac{x}{3}=24.$

3. $\frac{x}{4}+3x=26.$

4. $\frac{x}{2}+\frac{x}{3}=40.$

5. $\frac{2x}{5}-14=\frac{x}{10}+4.$

6. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=13.$

7. $\frac{x}{4}-\frac{x}{5}=13.$

8. $\frac{2x}{3}-\frac{3x}{7}=15.$

9. $\frac{x}{3}-14=\frac{x}{4}+5.$

10. $x+\frac{x}{3}+\frac{x}{4}=38.$

11. $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=94.$

12. $2x+\frac{x}{2}+\frac{x}{3}=34.$

13. $3x-\frac{2x}{5}+\frac{x}{7}=96.$

14. $\frac{x}{4}-x+\frac{x}{5}=-33.$

15. $2x=\frac{x}{7}-\frac{x}{3}+92.$

16. $\frac{x}{5}+\frac{x}{6}+\frac{x}{7}=214.$

17. $3x-\frac{2}{3}-\frac{x}{5}=41\frac{1}{8}.$

18. $x+\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=25.$

19. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=11.$

20. $\frac{1}{12}-\frac{1}{x}+\frac{1}{3x}=\frac{1}{4}-\frac{1}{4x}.$

21. $\frac{3x}{5}+\frac{5x}{6}-\frac{2x}{3}=15\frac{1}{8}.$

22. $\frac{2x}{7}-\frac{x-6}{4}=4-\frac{2x}{14}.$

23. $\frac{3-x}{2}-\frac{4-x}{5}-\frac{x}{6}=4.$

24. $\frac{x+3}{2}-\frac{x-3}{3}=\frac{3x-5}{12}.$

25. $\frac{x+3}{2}+\frac{x}{3}=4-\frac{x-5}{4}.$

26. $\frac{x-2}{3}-\frac{x-3}{4}=\frac{x-4}{5}-\frac{x-5}{6}.$

PROBLEMS.

Solve the following problems:

27. What number increased by its $\frac{1}{5}$ gives a sum of 60?

28. Two thirds of a number minus one fourth of the number equals 15; what is the number?

29. A horse and carriage cost $280; what was the cost of each if the carriage cost $\frac{2}{5}$ as much as the horse?

30. What number is that whose $\frac{1}{6}$ exceeds its $\frac{1}{7}$ by 20?

31. One half of my age 5 years ago equals one third of what my age will be 13 years hence; what is my age?

32. A teacher spent $\frac{1}{2}$ of his salary for board and clothes, and $\frac{1}{3}$ of the remainder for other expenses; what was his salary if he saved $200 a year?

33. Find two consecutive integral numbers, such that $\frac{1}{6}$ of the less exceeds $\frac{1}{7}$ of the greater by 5.

34. The difference between two numbers is 10, and $\frac{3}{7}$ of the smaller equals $\frac{2}{5}$ of the larger; what are the numbers?

35. How far can a person ride in a coach going at the rate of 10 miles an hour, provided he walks back at the rate of 4 miles an hour and is gone 14 hours?

36. A boat, whose rate of sailing in still water is 12 miles an hour, descends a stream whose current is 3 miles an hour; how far can it sail and be gone but 16 hours?

37. How far may a person ride on his bicycle, going at the rate of 12 miles an hour, provided he rides back in a car at the rate of 30 miles an hour and is gone but 7 hours?

38. A has a certain number of sheep which he wishes to put in pens, having the same number in each pen: if he put 8 sheep in a pen, it will require 6 pens more than if he put 10 sheep in a pen; how many sheep has he?

39. There is a number consisting of two digits: one third of the first digit equals the second; and the order of the digits becomes inverted by adding 36 to the number. What is the number?

ADDITION AND SUBTRACTION OF FRACTIONS.

125. Principle.—*To add or subtract fractions, they must have a common denominator.*

It is evident that $\frac{a}{m}+\frac{b}{m}=\frac{a+b}{m}$.

And $\frac{a}{m}-\frac{b}{m}=\frac{a-b}{m}$.

1. Add $\frac{4m-3n}{3mn^2}$ and $\frac{12m+3n}{4m^2n}$.

We reduce the fractions to their lowest common denominator (Prin.). The L. C. D. is $12m^2n^2$ (Art. 112). Hence

$$\begin{aligned}\frac{4m-3n}{3mn^2}+\frac{12m+3n}{4m^2n}&=\frac{16m^2-12mn}{12m^2n^2}+\frac{36mn+9n^2}{12m^2n^2}\\&=\frac{16m^2+24mn+9n^2}{12m^2n^2}\\&=\frac{(4m+3n)^2}{12m^2n^2}, \text{ Ans.}\end{aligned}$$

2. From $\frac{4x+7}{x-1}$ take $\frac{3x-2}{x+1}$.

The L. C. D. is x^2-1. Hence

$$\begin{aligned}\frac{4x+7}{x-1}-\frac{3x-2}{x+1}&=\frac{4x^2+11x+7}{x^2-1}-\frac{3x^2-5x+2}{x^2-1}\\&=\frac{4x^2+11x+7-(3x^2-5x+2)}{x^2-1}\\&=\frac{4x^2+11x+7-3x^2+5x-2}{x^2-1}\\&=\frac{x^2+16x+5}{x^2-1}, \text{ Ans.}\end{aligned}$$

Therefore, to add fractions,

RULE.

Reduce the fractions to their lowest common denominator, add their numerators, and write the result over the common denominator.

To subtract fractions,

RULE.

Reduce the fractions to their lowest common denominator, subtract the numerator of the subtrahend from the numerator of the minuend, and write the result over the common denominator.

NOTE.—**Always reduce the result to its simplest form.**

EXAMPLES.

Add the following:

3. $\frac{a}{bc}$, $\frac{b}{ac}$, and $\frac{c}{ab}$.

4. $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$.

5. $\frac{1}{m+n}$ and $\frac{1}{m-n}$.

6. $\frac{a}{a+b}$ and $\frac{b}{a-b}$.

7. $\frac{x-y}{x+y}$ and $\frac{x+y}{x-y}$.

8. $\frac{1+x^2}{1-x^2}$ and $\frac{1-x^2}{1+x^2}$.

9. $\frac{a}{a^2-b^2}$ and $\frac{b}{a^2+b^2}$.

10. $\frac{m}{m^2-n^2}$ and $\frac{n}{m+n}$.

11. $\frac{1}{x+x^2}$ and $\frac{1}{x-x^2}$.

12. $\frac{1+x}{1+x+x^2}$ and $\frac{1-x}{1-x+x^2}$.

13. $\frac{x}{(x-1)(x-2)}$ and $\frac{1}{(x-1)(2-x)}$.

14. $\frac{a}{(a-b)(b-c)}$ and $\frac{b}{(a-b)(c-b)}$.

Subtract the following:

15. $\frac{b}{ac}$ from $\frac{a}{bc}$.

16. $\frac{1}{xy}$ from $\frac{1}{yz}$.

17. $\frac{n}{m-n}$ from $\frac{m}{m+n}$.

18. $\frac{(c-d)^2}{cd}$ from $\frac{(c+d)^2}{cd}$.

19. $\frac{a-b}{a+b}$ from $\frac{a+b}{a-b}$.

20. $\frac{1}{m-n}$ from $\frac{1}{m+n}$.

21. $\dfrac{1-x^3}{1+x^3}$ from $\dfrac{1+x^3}{1-x^3}$.

22. $\dfrac{1+x^3}{1-x^3}$ from $\dfrac{1+x^2}{1-x^2}$.

23. $\dfrac{1}{m-m^2}$ from $\dfrac{1}{m+m^2}$.

24. $\dfrac{y}{x-y}$ from $\dfrac{x}{x^2-y^2}$.

25. $\dfrac{2}{(x-2)(x-3)}$ from $\dfrac{x}{(x-2)(3-x)}$.

26. $\dfrac{b}{(a-b)(b-c)}$ from $\dfrac{a}{(a-b)(c-b)}$.

Find the value of the following:

27. $\dfrac{1}{2+x}+\dfrac{2}{3-x}$.

28. $\dfrac{1}{x^2-xy}-\dfrac{1}{x^2-y^2}$.

29. $\dfrac{1}{x+7}-\dfrac{1}{x+11}$.

30. $\left(x-\dfrac{1}{x}\right)-\left(y-\dfrac{1}{x}\right)$.

31. $\dfrac{a-b}{ab}+\dfrac{b-c}{bc}$.

32. $\left(3a-\dfrac{b}{ax}\right)-\left(2a-\dfrac{a}{bx}\right)$.

33. $\left(4m-\dfrac{2n}{x}\right)+\left(m-\dfrac{n}{x}\right)$.

34. $\dfrac{(a+b)^2}{a+b}-\dfrac{(a-b)^2}{a+b}$.

35. $\dfrac{4}{x-5}-\dfrac{40}{x^2-25}$.

36. $\dfrac{60}{x^2-4}+\dfrac{15}{x+2}$.

37. $\dfrac{5}{x-2}-\dfrac{3}{x^3-8}$.

38. $\dfrac{x^4-x^2a^2+a^4}{x^5+a^5}-\dfrac{1}{x+a}$.

39. $\dfrac{1}{x^2-11x+30}-\dfrac{3}{x^2-13x+40}$.

40. $\dfrac{3}{x^2-1}+\dfrac{1}{x+1}-\dfrac{1}{x-1}$.

41. $\dfrac{a^2}{a^3-b^3}-\dfrac{b}{a^2+ab+b^2}+\dfrac{b}{a-b}$.

42. $\dfrac{2x^3+2x}{x^4-b^4}-\left(\dfrac{1}{x+b}+\dfrac{1}{x-b}\right)$.

43. $\frac{1}{6} - \frac{-x}{x^2 - x - 6} + \frac{-2}{(3 - x)(x + 2)}.$

44. $\frac{x + 1}{x^3 + 1} - \frac{x - 1}{x^3 - 1} - \frac{x^2}{x^5 + x^3 + x}.$

45. $\frac{2a}{a^4 + a^2b^2 + b^4} + \frac{1}{a^2 + ab + b^2} - \frac{1}{a^2 - ab + b^2}.$

46. $\frac{1}{x^2 - 7x + 12} + \frac{2}{x^2 - 4x + 3} + \frac{3}{x^2 - 5x + 4}.$

47. $\frac{1}{n + 1} + \frac{2}{n + 2} + \frac{3}{n + 3}.$

48. $\frac{1}{x + 1} - \frac{2}{x - 2} + \frac{3}{x - 3}.$

49. $\frac{x - 5}{x^2 - 4} + \frac{8}{x + 2} - \frac{3}{x - 2}.$

50. $\frac{x + 6}{x^2 + 9x + 20} - \frac{x + 5}{x^2 + 10x + 24}.$

51. $\frac{x - 3}{x^2 - 3x + 2} - \frac{x - 2}{x^2 - 4x + 3} - \frac{x - 1}{x^2 - 5x + 6}.$

52. $\frac{x^2 - x + 1}{x^3 + 1} - \frac{x^3 + x^2 + x + 1}{x^4 - 1} + \frac{3x^4 + 3x^2 + 3}{x^6 - 1}.$

53. $\frac{1}{x^2 - x + 1} - \frac{2x^2}{x^4 + x^2 + 1} + \frac{1}{x^2 + x + 1}.$

54. $\frac{x - 2}{x^2 - x + 1} + \frac{x^2 - 2x + 3}{x^3 + 1} - \frac{1}{x + 1}.$

55. $\frac{1 + x}{2x^3 - x^2 - 2x + 1} - \frac{3x^2}{x + x^2 - 6x^3} + \frac{2x^2 - 4x^3}{1 - 7x^2 + 6x^3}.$

56. $\frac{l}{(l - m)(l - n)} + \frac{m}{(m - n)(m - l)} - \frac{-n}{(n - l)(n - m)}.$

57. $\frac{a^2 + 7a + 12}{a^3 + 3a^2 - 9a - 27} + \frac{a^2 - 2a - 15}{a^3 - 5a^2 - 9a + 45} - \frac{a^2 - 16}{a^3 - 4a^2 - 9a + 36}.$

MULTIPLICATION OF FRACTIONS.

CASE I.

126. To multiply a fraction by an integer.

1. Multiply $\frac{a}{b}$ by c, and $\frac{a}{bc}$ by c.

Since multiplying the numerator or dividing the denominator multiplies the value of the fraction (Art. 119, Prin. 1 and 4), then

$$\frac{a}{b} \times c = \frac{ac}{b}.$$

And $$\frac{a}{bc} \times c = \frac{a}{b}.$$

Therefore, to multiply a fraction by an integer,

RULE.

Multiply the numerator or divide the denominator of the fraction by the multiplier.

Since the product is the same in whatever order the factors are taken (Art. 71, Prin. 1), then

$$\frac{a}{b} \times c = c \times \frac{a}{b}.$$

It is frequently convenient to indicate the multiplication, and cancel common factors from both terms.

2. Multiply $\frac{m+n}{m^2-2mn+n^2}$ by (m^2-n^2).

$$\frac{m+n}{m^2-2mn+n^2} \times (m^2-n^2) = \frac{m+n}{(m-n)(m-n)} \times (m-n)(m+n)$$

$$= \frac{(m+n)(m+n)}{m-n}$$

$$= \frac{(m+n)^2}{m-n}, \textit{ Ans.}$$

EXAMPLES.

Multiply the following:

3. $\frac{m}{n}$ by a.
4. $\frac{ab}{c^2d}$ by ac.
5. $\frac{2x^2}{3n^3}$ by $4x^2$.
6. $\frac{5m^2}{12x^2n}$ by $4x$.
7. $\frac{8xy}{15m^2z}$ by $5xm$.
8. $\frac{4an}{21m^3x^2y}$ by $7am^2x^2$.

9. $\dfrac{4ay}{(a-y)^2}$ by a^2-y^2.

10. $\dfrac{ayx}{x^6+y^6}$ by $x^4-x^2y^2+y^4$.

11. $\dfrac{5mn^2}{m^4-n^4}$ by m^2-n^2.

12. $\dfrac{4x^2}{x^4-x}$ by x^2+x+1.

13. $\dfrac{6m^2n}{5(m^2-25)}$ by $5(m+5)$.

14. $\dfrac{3ay}{x^3y-xy^3}$ by $ax+ay$.

15. x^2-1 by $\dfrac{x^2-1}{x^2+1}$.

16. $(x+1)^3$ by $\dfrac{x^2-x+1}{x^3+1}$.

17. y^3-y^2 by $\dfrac{y}{y^2-1}$.

18. $\dfrac{4xy}{x^3+x^2+x}$ by x^3-1.

19. $\dfrac{4n}{x^2+x+1}$ by x^4+x^2+1.

20. $x^2+7x+12$ by $\dfrac{x+3}{x+4}$.

21. $\dfrac{x+12}{x-11}$ by x^3+x^2-132x.

22. $\dfrac{a-y}{a^2-x^2}$ by $a^2+2ax+x^2$.

CASE II.

127. To multiply a fraction by a fraction.

1. Multiply $\dfrac{a}{b}$ by $\dfrac{c}{d}$.

$\dfrac{a}{b}$ multiplied by c is $\dfrac{ac}{b}$ (Case I.); but we wish to multiply by $\dfrac{c}{d}$. Since the multiplier is divided by d, the product, $\dfrac{ac}{b}$, must be divided by d, which by ART. 119, Prin. 3, is $\dfrac{ac}{bd}$.

$$\text{Therefore,}\quad \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd},\ \textit{Ans.}$$

2. Multiply together $\dfrac{m^2-4}{m^2-mn}$, $\dfrac{m^2+m}{m^2-m-6}$, and $\dfrac{m^2-3m}{m^2-m-2}$.

$$\text{This} = \frac{(m+2)(m-2)}{m(m-n)}\times\frac{m(m+1)}{(m-3)(m+2)}\times\frac{m(m-3)}{(m+1)(m-2)}$$

$$=\frac{m}{m-n},\ \textit{Ans.}$$

Therefore, to multiply a fraction by a fraction,

RULE.

Multiply the numerators together for the numerator, and the denominators together for the denominator, canceling common factors.

EXAMPLES.

Multiply the following:

3. $\dfrac{4x^2y^2}{5z^2y}$ by $\dfrac{20z^2}{16xy}$.

4. $\dfrac{5a^2bc^2}{3x^2y^2z}$ by $\dfrac{9x^2y^2z}{5abc}$.

5. $\dfrac{a+c}{c^2}$ by $\dfrac{ac+cx}{a^2-x^2}$.

6. $\dfrac{(a-b)^2}{a+b}$ by $\dfrac{(a+b)^2}{a-b}$.

7. $\dfrac{a^2+ab}{(1+b)^2}$ by $\dfrac{c+bc}{a+b}$.

8. $\dfrac{a(a-b)}{(a+b)^2}$ by $\dfrac{a^2-b^2}{(a-b)^2}$.

9. $\dfrac{a^3+b^3}{a^3-b^3}$ by $\dfrac{a^2+ab+b^2}{a^2-ab+b^2}$.

10. $\dfrac{a^2-b^2}{a^2+b^2}$, $\dfrac{a^4-b^4}{(a+b)^2}$, and $\dfrac{a+b}{(a-b)^2}$.

11. $\dfrac{a^2+2ab+b^2-c^2}{a^2-2ab+b^2-c^2}$ by $\dfrac{a-b+c}{a+b+c}$.

12. $\dfrac{a^4-y^4}{a^2+2ab+b^2}$ by $\dfrac{a+b}{x^3-x^2y+xy^2-y^3}$.

13. $\dfrac{x^2-y^2-z^2-2yz}{x^2-xy-xz}$ by $1-\dfrac{2z}{x+y+z}$.

14. $\dfrac{x^2-5x-14}{x^2-12x+27}$ by $\dfrac{x^2-11x+18}{x^2-4x-21}$.

15. $\dfrac{x^6-y^6}{x^6+y^6}$, $\dfrac{x-y}{x^4-y^4}$, and $\dfrac{x^2+y^2}{x^3-y^3}$.

16. $\left(\dfrac{a^2}{b-c}+\dfrac{b^2}{c-b}\right)$ by $\left(\dfrac{b}{a^2-b^2}+\dfrac{c}{b^2-a^2}\right)$.

17. $\left\{a - \frac{a^2+b^2}{a}\right\}$ by $\left\{a + \frac{a^2+b^2}{a}\right\}$.

18. $\left\{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right\}$ by $\frac{xyz}{x^2+y^2-z^2+2xy}$.

19. $\frac{x^6-y^6}{x^4+x^2y^2+y^4}$, $\frac{x^2+y^2}{x^2-xy+y^2}$, and $\frac{x+y}{x^3-y^3}$.

20. $\frac{a^4-b^4}{(a-b)^2}$, $\frac{a-b}{a^2+ab}$, and $\frac{1}{a^2+b^2}$.

21. $\frac{x^2-10x+21}{x^2+4x-45}$ by $\frac{x^2+2x-35}{x^2-14x+49}$.

22. $\frac{m^{3x}-n^{6y}}{m^{2x}-n^{2y}}$ by $\frac{m^x-n^y}{m^{2x}+m^xn^{2y}+n^{4y}}$.

DIVISION OF FRACTIONS.

CASE I.

128. To divide a fraction by an integer.

1. Divide $\frac{ac}{b}$ by c, and $\frac{a}{b}$ by c.

Since dividing the numerator or multiplying the denominator divides the value of the fraction (ART. 119, Prin. 2 and 3), then

$$\frac{ac}{b} \div c = \frac{a}{b}.$$

And $$\frac{a}{b} \div c = \frac{a}{bc}.$$

Therefore, to divide a fraction by an integer,

RULE.

Divide the numerator or multiply the denominator of the fraction by the divisor.

It is frequently convenient to indicate the division, and cancel common factors from both terms.

2. Divide $\frac{n^2-16}{n+3}$ by n^2+n-12.

$$\frac{n^2-16}{n+3} \div n^2+n-12 = \frac{(n+4)(n-4)}{(n+3)\times\{(n+4)(n-3)\}}$$

$$= \frac{n-4}{n^2-9}, \textit{ Ans.}$$

EXAMPLES.

Divide the following:

3. $\frac{x}{y}$ by n.

4. $\frac{12m^2n}{5xy}$ by $6mn$.

5. $\frac{12x^2y}{25ab^2}$ by $6x^2b^2$.

6. $\frac{ab^2c}{mn}$ by ac.

7. $\frac{7ax}{5mn}$ by $5mn$.

8. $\frac{18n^2x^3}{5m^2y^3}$ by $9x^3y^3$.

9. $\frac{(a-x)^2}{4ay}$ by a^2-x^2.

10. $\frac{x^3-y^3}{3ax}$ by x^2+xy+y^2.

11. $\frac{m^2n-n^3}{4an}$ by $m-n$.

12. $\frac{ca^3-ab^2c}{a^2-ac}$ by $ac-bc$.

13. $\frac{a^2+1}{a-1}$ by $a-1$.

14. $\frac{x^5-x^2}{11mx}$ by x^2+x+1.

15. $\frac{a^{2n}-b^{2n}}{a-b}$ by a^n-b^n.

16. $\frac{m^4n-mn^4}{4mn}$ by a^nm-a^nn.

17. $\frac{n^2+7n-30}{n+5}$ by $n^2+15n+50$.

18. $\frac{x^2+(a-b)x-ab}{a+b}$ by $x-b$.

19. $\frac{x^4+x^2+1}{x-1}$ by x^2-x+1.

20. $\frac{x^3+2x^2+x}{x-1}$ by $(x+1)^2$.

21. $\frac{x^5y-xy^5}{x^2y^2}$ by $x^3+y^3+x^2y+xy^2$.

CASE II.

129. To divide a fraction by a fraction.

1. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

$\frac{a}{b}$ divided by c is $\frac{a}{bc}$ (Case I.); but we wish to divide by $\frac{c}{d}$. Since the divisor is *divided* by d, the quotient, $\frac{a}{bc}$, must be *multiplied* by d, which by Art. 119, Prin. 1, gives $\frac{ad}{bc}$.

Therefore, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$, *Ans.*

2. Divide $\frac{m^2 - 4}{m^2 - mn}$ by $\frac{m^2 + 3m - 10}{mn - n^2}$.

$$\frac{m^2 - 4}{m^2 - mn} \div \frac{m^2 + 3m - 10}{mn - n^2} = \frac{(m + 2)(m - 2)}{m(m - n)} \times \frac{n(m - n)}{(m - 2)(m + 5)}$$
$$= \frac{n(m + 2)}{m(m + 5)}, \text{ Ans.}$$

Therefore, to divide a fraction by a fraction,

RULE.

Invert the divisor, and proceed as in multiplication.

If the dividend is an integer, it may be considered as a fraction whose denominator is 1; as, $m + n = \frac{m + n}{1}$.

EXAMPLES.

Divide the following:

3. $\frac{6x^2y^2}{8ab^2}$ by $\frac{3xy^2}{4a^2b}$.

4. $\frac{14m^3n^2}{9x^2y^3}$ by $\frac{7m^2n^2}{3x^2y^2}$.

5. $21am^2$ by $\frac{7am}{n}$.

6. $\frac{x^2 - 1}{ab}$ by $\frac{x + 1}{b}$.

7. $1 - \frac{1}{x^2}$ by $1 + \frac{1}{x}$.

8. $x^3 - \frac{1}{x^3}$ by $x^2 + 1 + \frac{1}{x^2}$.

9. $\dfrac{(m-n)^2}{m+n}$ by $\dfrac{m-n}{(m+n)^2}$.

10. $\dfrac{mx^2-my^2}{(m+1)^2}$ by $\dfrac{m^2x-m^2y}{am+a}$.

11. $\dfrac{m^3-n^3}{m^3+n^3}$ by $\dfrac{m-n}{m+n}$.

12. $\dfrac{a^2}{b^2}-\dfrac{c^2}{d^2}$ by $\dfrac{a}{b}+\dfrac{c}{d}$.

13. $\dfrac{x^6-y^6}{x^6+y^6}$ by $\dfrac{x^4+x^2y^2+y^4}{x^4-x^2y^2+y^4}$.

14. $\dfrac{a^2}{b^2}+2+\dfrac{b^2}{a^2}$ by $\dfrac{a}{b}+\dfrac{b}{a}$.

15. $m^5+\dfrac{1}{m^5}$ by $m+\dfrac{1}{m}$.

16. $x^{12}+\dfrac{1}{x^{12}}$ by $x^4+\dfrac{1}{x^4}$.

17. $\dfrac{x^2+10x-119}{x^2-40x+319}$ by $\dfrac{x^2+20x+51}{x^2-20x-261}$.

18. $\dfrac{x^2}{x+y}+\dfrac{xy}{x-y}$ by $\dfrac{x}{x-y}+\dfrac{y}{x-y}$.

19. $\dfrac{m^2+3m-40}{m^2-3m-40}$ by $\dfrac{m^2-12m+35}{m^2-15m+56}$.

20. $\dfrac{n^2-16n+60}{n^2+14n+45}$ by $\dfrac{n^2-2n-80}{n^2+n-72}$.

21. $\dfrac{x^2+2xy+y^2-z^2}{y^2-x^2-2xz-z^2}$ by $\dfrac{x^2-(z-y)^2}{(x+z)^2-y^2}$.

22. $\dfrac{(x+2)^2-9}{(x+3)^2-16}$ by $\dfrac{(x+1)^2-16}{(x+5)^2-4}$.

23. $1+\dfrac{y}{x+y}+\dfrac{x}{y}$ by $2+\dfrac{x}{y}-\dfrac{x}{x+y}$.

24. $\left(x+\dfrac{xy}{x-y}\right)\left(x-\dfrac{xy}{x+y}\right)$ by $\left(\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}\right)$.

25. $\left(\dfrac{x^2}{a^2}+\dfrac{a^2}{x^2}-\dfrac{x}{a}-\dfrac{a}{x}+2\right)$ by $\left(\dfrac{x}{a}+\dfrac{a}{x}\right)$.

26. $\left(\dfrac{1}{(m-n)^2}-\dfrac{1}{a^2}\right)$ by $\left(\dfrac{m-n+a}{m^2-n^2}\right)$.

27. $\left(\dfrac{x}{x-y}-\dfrac{y}{x+y}\right)$ by $\left(\dfrac{x^2}{x^2+y^2}+\dfrac{y^2}{x^2-y^2}\right)$.

COMPLEX FRACTIONS.

130. A **Complex Fraction** is one whose numerator or denominator, or both, are fractional.

A complex fraction may be regarded as a case in division.

1. Reduce $\dfrac{\frac{m}{n}}{\frac{x}{y}}$ to a simple fraction.

Regarding this as a case in division, and dividing $\frac{m}{n}$ by $\frac{x}{y}$, we have, by Art. 129, $\frac{my}{nx}$.

Or, we may multiply both terms of the fraction by ny, the L. C. M. of the denominators (Art. 119, Prin. 5), and we will obtain $\frac{my}{nx}$.

1st Process.

$$\frac{\frac{m}{n}}{\frac{x}{y}} = \frac{m}{n} \div \frac{x}{y} = \frac{m}{n} \times \frac{y}{x} = \frac{my}{nx}, \textit{ Ans.}$$

2d Process.

$$\frac{\frac{m}{n} \times ny}{\frac{x}{y} \times ny} = \frac{my}{nx}, \textit{ Ans.}$$

2. Reduce $\dfrac{\frac{m}{m+n} + \frac{n}{m-n}}{\frac{m}{m-n} - \frac{n}{m+n}}$ to a simple fraction.

The L. C. M. of $m + n$ and $m - n$ is $m^2 - n^2$. Multiplying both terms of the fraction by $m^2 - n^2$, we have

$$\frac{m(m-n) + n(m+n)}{m(m+n) - n(m-n)} = \frac{m^2 + n^2}{m^2 + n^2} = 1, \textit{ Ans.}$$

EXAMPLES.

Reduce the following to simple fractions:

3. $\dfrac{1 - \frac{1}{a}}{1 + \frac{1}{a}}$.

4. $\dfrac{a + \frac{b}{c}}{a - \frac{b}{c}}$.

5. $\dfrac{\frac{a}{b} - \frac{b}{a}}{\frac{a+b}{ab}}$.

6. $$\frac{\dfrac{b^2}{a}-a}{\dfrac{b^2}{a}+a}.$$

7. $$\frac{\dfrac{a-b}{ab}}{\dfrac{1}{a}+\dfrac{1}{b}}.$$

8. $$\frac{m+4-\dfrac{2}{m}}{m-2+\dfrac{3}{m}}.$$

9. $$\frac{\dfrac{x}{x-1}-1}{1-\dfrac{x}{x+1}}.$$

10. $$\frac{a+b-\dfrac{4}{a}}{a-b+\dfrac{6}{a}}.$$

11. $$\frac{x}{x+\dfrac{1}{x+\dfrac{1}{x+1}}}.$$

12. $$\frac{\dfrac{y}{x-y}+\dfrac{x}{x+y}}{\dfrac{y}{x+y}-\dfrac{x}{x-y}}.$$

13. $$\frac{1}{m+\dfrac{1}{m+\dfrac{1}{m+1}}}.$$

14. $$\frac{m-6+\dfrac{3}{m-2}}{m+8-\dfrac{39}{m-2}}.$$

15. $$\frac{a+6-\dfrac{3}{a+4}}{a+1-\dfrac{18}{a+4}}.$$

16. $$\frac{1}{m-\dfrac{1}{m-\dfrac{1}{m-1}}}.$$

17. $$\frac{\dfrac{2}{3-n}+6-n}{\dfrac{24}{3-n}-7-n}.$$

18. $$\frac{ab}{a+b-\dfrac{1}{a-b+\dfrac{b^2}{a+b}}}.$$

19. $$\frac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{x}{x-y}-\dfrac{y}{x+y}} \div \frac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{x}{x+y}+\dfrac{y}{x-y}}.$$

20. $$\frac{\dfrac{1}{(m+n)(m-l)}+\dfrac{1}{(m-n)(m+l)}}{\dfrac{1}{(m+n)(m+l)}+\dfrac{1}{(m-n)(m-l)}}.$$

21. $$\left[\frac{a}{1+\dfrac{a}{1+a}}+1-\frac{a}{1-\dfrac{a}{1+a}}\right]\div\left[\frac{a}{1-\dfrac{a}{1+a}}\right].$$

22. $$\frac{\dfrac{x}{\dfrac{1}{x-1}+x}}{1-\dfrac{x}{x-\dfrac{1}{1-\dfrac{x}{x-1}}}}.$$

23. $$\frac{m+\dfrac{m-\dfrac{m}{2}}{2}}{m+\dfrac{2}{m-\dfrac{2}{m}}}.$$

24. $$\frac{\dfrac{x+2}{x-3}-\dfrac{x+3}{x-2}}{\dfrac{x-2}{x+3}-\dfrac{x-3}{x+2}}\times\frac{\dfrac{x+1}{x+3}+\dfrac{x-1}{x+2}}{\dfrac{x+2}{x-3}+\dfrac{x-3}{x-2}}.$$

REVIEW.

1. Reduce $\dfrac{15x^2+11x-14}{18x^2+9x-14}$ to its lowest terms.

2. Reduce $\dfrac{a^{12}+b^{12}}{a^6+b^6}$ to a mixed quantity.

3. Reduce $m^2+3m+1-\dfrac{5m^2+7m}{m+2}$ to a fraction.

4. Reduce $\dfrac{x-2}{x+3}$, $\dfrac{x+4}{3x+9}$, and $\dfrac{x-5}{4x+12}$ to a common denominator.

5. Add $\dfrac{b}{(a-b)(a-c)}$ and $\dfrac{a}{(b-a)(b-c)}$.

6. Subtract $\dfrac{1}{(a-b)\ (b-c)}+\dfrac{1}{(c-b)\ (c-a)}$ from $\dfrac{1}{(a-b)(a-c)}$.

7. Multiply $\frac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc}$ by $\frac{a-b+c}{a-b-c}$.

8. Divide $\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)$ by $\left(\frac{x-y}{x+y}+\frac{x+y}{x-y}\right)$.

9. Reduce $\frac{x^4-x^3-x+1}{x^4-x^3-x^2+x}$ to its lowest terms.

10. Reduce $\frac{1+x+x^4-x^5}{1-x}$ to a mixed quantity.

11. Reduce $x^8-x^2y^2+y^8-\frac{x^{12}+y^{12}}{x^4+y^4}$ to a fraction.

12. Add $\frac{1}{a(a-b)(a-c)}$, $\frac{1}{b(b-a)(b-c)}$, and $\frac{-1}{abc}$.

13. Subtract $\frac{x^2-3x-28}{x^2+x-12}$ from $\frac{x^2-8x-33}{x^2-4x-77}$.

14. Multiply $\left[\frac{(a^3+b^3)^n}{(a^5-b^5)^n}\times\frac{(a^{10}-b^{10})^n}{(a^6-b^6)^n}\right]$ by $\frac{(a-b)^n}{(a+b)^n}$.

15. Divide $\left(1+\frac{x}{y}\right)\left(1+\frac{z}{y}\right)$ by $\left(1+\frac{y}{x}\right)\left(1+\frac{y}{z}\right)$.

16. Reduce $\frac{(a-b)^5-a^5+b^5}{(a-b)^4-a^4+b^4}$ to its lowest terms.

17. Reduce $\frac{(x^2+y^2)^3-x^6-y^6}{(x^2+y^2)^2-x^4+y^4}$ to its lowest terms.

18. Reduce $\frac{2x^5}{x-1}$ to a mixed quantity.

19. Reduce $\frac{2y^{12}}{y^4+x^4}$ to a mixed quantity.

20. Reduce $x^2+xy+y^2-\frac{x^4-x^2y^2+y^4}{x^2-xy+y^2}$ to a fraction.

21. Reduce $x^8-x^6y^2+x^4y^4-x^2y^6+y^8+\frac{x^{10}+y^{10}}{x^2+y^2}$ to its simplest form.

22. Add $\frac{1}{x-1}$, $\frac{1}{x^3+1}$, $\frac{1}{x^3-1}$, and $\frac{1}{x+1}$.

23. Add $\frac{2x+1}{x^2+1}$ and $\frac{2x+1}{x^2-1}$.

24. Add $\frac{x+1}{x^2+x+1}$ and $\frac{x+1}{x^2-x+1}$.

25. Subtract $\frac{3x+2}{x^3+1}$ from $\frac{3x+2}{3x-1}$.

26. Subtract $\frac{x^2-a^2}{x^3+a^3}$ from $\frac{x^2-a^2}{x^3-a^3}$.

27. Multiply $\frac{8m^2-8m}{7m^4+14m^2+7}$ by $\frac{9m^2n+9n}{10m^2-20m+10}$.

28. Divide $(a+b)^7-a^7-b^7$ by $(a+b)^5-a^5-b^5$.

29. Expand $\left(\frac{x}{y}+1\right)^2$ and $\left(\frac{x}{y}-1\right)^2$.

30. Expand $\left(\frac{a}{b}+\frac{b}{a}\right)^2$ and $\left(\frac{a}{b}-\frac{b}{a}\right)^2$.

31. Expand $\left(\frac{m}{n}+1\right)\left(\frac{m}{n}-1\right)$.

32. Expand $\left(\frac{m}{n}+\frac{a}{b}\right)\left(\frac{m}{n}+\frac{c}{d}\right)$.

33. Factor $a^6+\frac{y^6}{x^6}$ and $\frac{m^3}{n^3}-x^3$.

34. Factor $a^2+2+\frac{1}{a^2}$ and $a^2+1+\frac{1}{a^2}$.

35. Factor $\frac{(a+b)^2-(a-b)^2}{(c+d)^2-(c-d)^2}$ and $\frac{(a+b)^2}{(c+d)^2}-\frac{(a-b)^2}{(c-d)^2}$.

36. Factor $a^2-\left(\frac{a^2+b^2-c^2}{2b}\right)^2$;

37. Factor $\frac{m^2}{n^2} - \frac{m}{n} - 30$ and $\left(\frac{x}{y}\right)^2 - 13\frac{x}{y} + 30$.

38. If $s = a + b + c$, show that $\left(\frac{a+b+c}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a-b+c}{2}\right)$ $\left(\frac{c-a+b}{2}\right) = \frac{1}{2}s\left(\frac{1}{2}s - a\right)\left(\frac{1}{2}s - b\right)\left(\frac{1}{2}s - c\right)$.

39. Reduce $\frac{x^4 + x^3 + x^2 + x + 1}{x^5 - 1}$ to its lowest terms.

40. Add $\frac{x-y}{xy}$, $\frac{z-x}{xz}$, and $\frac{y-z}{yz}$.

41. Reduce $\frac{a^{12} + y^{12}}{a^{20} + y^{20}}$ to its lowest terms.

42. Divide $\frac{x^2}{y^2} + 1 + \frac{y^2}{x^2}$ by $\frac{x}{y} - 1 + \frac{y}{x}$.

43. Multiply $\frac{x^2 - x - 6}{x^2 + 5x + 6}$ by $\frac{x^2 + 8x + 15}{x^2 + 2x - 15}$.

44. Subtract $\frac{(x-1)^4}{x}$ from $(x-1)^4$.

45. Simplify $$\frac{\dfrac{(a-b)^2 - c^2}{a^2 - (b-c)^2}}{\dfrac{(a-b-c)(a+b-c)}{(a+b-c)(a-b+c)}}.$$

46. Simplify $$\frac{1 + \dfrac{a-c}{a+c}}{1 - \dfrac{a-c}{a+c}} \div \frac{1 + \dfrac{a^2 - c^2}{a^2 + c^2}}{1 - \dfrac{a^2 - c^2}{a^2 + e^2}}.$$

47. Simplify $$\frac{\dfrac{(a+b)^3}{x^3 - y^3} \times \dfrac{x^2 + xy + y^2}{(a+b)^2}}{\dfrac{(x^2 - y^2)(a^2 - b^2)}{(a^2 - 2ab + b^2) \div (x^2 - 2xy + y^2)}}.$$

VI. SIMPLE EQUATIONS.

DEFINITIONS.

131. Review the following definitions:

1. Of an **Equation** (Art. 10).
2. Of the **Members of an Equation** (Art. 66 and 67).
3. Of **Transposition** (Art. 68).
4. Of **Clearing of Fractions** (Art. 124).
5. Of **Numerical Equations** (Art. 74).
6. Of **Literal Equations** (Art. 75).
7. Of **Identical Equations** (Art. 76).
8. Of the **Root of an Equation** (Art. 77).
9. Of an **Axiom** (Art. 54). State axiomatic principle (Art. 69).
10. Of a **Solution of an Equation** (Art. 59).
11. Of a **Problem** (Art. 60).

132. The **Degree** of an equation containing but one unknown quantity is determined by the highest power of the unknown quantity.

Thus, $x + a = 4$ is an equation of the *first* degree.

$2x^2 + 3x = 4$ is an equation of the *second* degree.

$5x^3 + 4x^2 + 3x = 2$ is an equation of the *third* degree.

$x^4 + 2x^3 + 3x^2 + 4x = 5$ is an equation of the *fourth* degree.

133. A **Simple Equation** is an equation of the *first* degree.

134. A **Quadratic Equation** is an equation of the *second* degree.

135. A **Cubic Equation** is an equation of the *third* degree.

136. A **Bi-quadratic Equation** is an equation of the *fourth* degree.

SOLUTION OF NUMERICAL EQUATIONS.

EXAMPLES.

1. $17x + 42 = 10x + 181.$

2. $102x - 45 = 90 + 98x.$

3. $110x + 82 = 120x - 28.$

4. $5x^2 + 7x = 2x^2 + 34x.$

5. $7x^2 - 8x + x^2 = 25x - 3x^2.$

6. $3x^2 + 5x + 2x^2 = x^2 + 17x.$

7. $5x + 18 - 2x = 21 + x + 3.$

8. $42 - 8x = 4x - 6 - 6x.$

9. $11x - 42 + 9x = -6x + 114.$

10. $110 - 19x + 45 = 11x - 21.$

11. $5(x - 2) - 6(x + 4) = 21.$

12. $9(x - 2) = 10(x - 1) - 12.$

13. $(x - 3)(x - 2) = x^2.$

14. $(x + 7)^2 - (x - 7)^2 + 2x = 900.$

15. $\dfrac{5x + 2}{4} - \dfrac{10x - 8}{5} - 6 = \dfrac{4}{x} + \dfrac{6x}{5} - \dfrac{39x}{20} + \dfrac{1}{10}.$

16. $\dfrac{1}{x + 1} + \dfrac{1}{x + 2} = \dfrac{1}{x^2 + 3x + 2}.$

17. $\dfrac{1}{x - 1} + \dfrac{1}{x - 2} = \dfrac{1}{x^2 - 3x + 2}.$

18. $\dfrac{1}{x - 2} = \dfrac{2}{x + 2} + \dfrac{1}{x^2 - 4}.$

19. $\dfrac{2}{x - 2} - \dfrac{3}{x - 3} + \dfrac{1}{x - 4} = 0.$

20. $\dfrac{1}{x - 2} - \dfrac{1}{x - 4} = \dfrac{1}{x - 6} - \dfrac{1}{x - 8}.$

21. $x - 2\{x - 4(x - 3) + 4\} = 12.$

22. $2[x - \{3 - x - (4 - \overline{x - 5}) - \overline{6 + x}\} - 7] = 48.$

23. $(3x + 1)(4x + 7) = 12(x + 1)^2.$

24. $(8x - 3)^2 (x - 1) = (4x - 1)^2 (4x - 5).$

25. $\dfrac{x + 1}{x - 1} - \dfrac{x - 1}{x + 1} = \dfrac{4}{x^2 - 1}.$

26. $x - 2 + \dfrac{x}{x + 2} + \dfrac{x + 1}{2} = x - \dfrac{2 - x}{2}.$

SOLUTION OF LITERAL EQUATIONS.

137.—1. Find the value of x in $\dfrac{x}{a} + \dfrac{x}{b} = c.$

Clearing the equation of fractions, we have

$$bx + ax = abc.$$

Factoring the first member, $(a + b)x = abc.$

Dividing by $a + b$, $x = \dfrac{abc}{a + b}$, *Ans.*

2. Find the value of x in $(x + a)^2 - (x - a)^2 = 8a.$

Expanding, $x^2 + 2ax + a^2 - x^2 + 2ax - a^2 = 8a.$

Uniting terms, $4ax = 8a.$

Dividing by $4a$, $x = 2$, *Ans.*

EXAMPLES.

Solve the following equations:

3. $ax + bx = an + bn.$

4. $ax + d = c - dx.$

5. $mx - n = nx + m.$

6. $lx + p = mx + q.$

7. $a(x - n) = n(a - x).$

8. $a(x+b-c)=c(n-x-a)$.

9. $4mx-na+x=3mx+an-3x$.

10. $(x-a)(x-b)=(x-a-b)^2+ab$.

11. $\dfrac{a+b}{a-x}+\dfrac{b+c}{c-x}=\dfrac{a-b}{a-x}$.

12. $(a+x)(b+x)-a(b+c)=\dfrac{a^2c}{b}+x^2$.

13. $\dfrac{x}{a}-b=\dfrac{m}{n}-x$.

14. $ax+b=\dfrac{x}{a}+\dfrac{1}{b}$.

15. $\dfrac{5x-a}{3x+a}-\dfrac{2x+a}{3x-a}=1$.

16. $\dfrac{x-\dfrac{x-c}{2}}{x+\dfrac{x-c}{3}}=1$.

17. $S=\left(\dfrac{x+l}{2}\right)n$.

18. $S=\dfrac{xl-a}{x-1}$.

19. $\dfrac{x}{a}-\dfrac{x}{b}=\dfrac{a}{b}-\dfrac{x}{ab}$.

20. $x+a=\dfrac{x^2}{a+x}$.

21. $\dfrac{x^3-a^3}{x-a}=x^2-bx+b^2$.

22. $\dfrac{a-\dfrac{x}{a}}{\dfrac{b}{n}+\dfrac{x}{b}}=n$.

23. $S=\left(\dfrac{a+l}{2}\right)x$.

24. $S=\dfrac{rx-a}{r-1}$.

25. $\dfrac{x^2-\dfrac{1}{x^2}}{x+\dfrac{1}{x}}+2abc+\dfrac{\dfrac{1}{x^2}-x^2}{x-\dfrac{1}{x}}=0$.

26. $\dfrac{nx-m}{nx+m}-\dfrac{mx-n}{mx+n}=\dfrac{n-m}{(nx+m)(mx+n)}$.

27. $\dfrac{x-a}{b}-\dfrac{x^2-bx-a^2}{bx-a^2}=\dfrac{bx}{bx-a^2}$.

MISCELLANEOUS EXAMPLES.

138. Solve the following equations:

1. $\frac{4x}{5} - 2\frac{1}{3} = \frac{3x}{10} + 3\frac{2}{3}$.

2. $\frac{a+x}{a-b} - \frac{a-x}{a+b} = \frac{a^2+b^2}{a^2-b^2}$.

3. $\frac{7x}{8} + \frac{5x}{4} - 2\frac{2}{5} = \frac{12x}{15} - 3\frac{4}{5} - \frac{10x}{32}$.

4. $\frac{2x}{m+n} - \frac{5x}{m-n} + \frac{3x}{m^2-n^2} = \frac{1}{7(n-m)}$.

5. $\frac{7-9x}{12} - \frac{12-4x}{5-3x} = \frac{15-6x}{8} - \frac{7}{24}$.

6. $cm - d - \frac{cx}{d} + \frac{x}{m} = 0$.

7. $\frac{10x-12}{15} - \frac{12-15x}{18} = \frac{16x-16}{24}$.

8. $\frac{x^2+3x+4}{x+7} = x - 12$.

9. $\frac{y^2+2ay+b^2}{y-b} = y - b + 2$.

10. $\frac{12x+5}{24} = \frac{50x-20}{25} - \frac{15x+6}{30}$.

11. $3ax - 2bx - \frac{1}{3}c - \frac{1}{4}mx = \frac{2}{3}c + \frac{3}{4}mx - n - bx + 2ax$.

12. $(m+n)x - \frac{2mx}{m-n} = (m-n)x - \frac{4mn}{m-n}$.

13. $\frac{x-6}{9} - \frac{15+4x}{7} = \frac{6x}{14} - \frac{5-7x}{2-3x} - \frac{24x-9}{27}$.

14. $5\left(\frac{5x+6}{3}\right) - \left(10x - \frac{9x-6}{6}\right) = 7$.

15. $m^2x - m^2 = n^2 - 2mn + n^2x.$

16. $(a - x)^2 - (b - x)^2 = (a - b)^2.$

17. $\dfrac{my^2}{n - cy} + m + \dfrac{my}{c} = 0.$

18. $\dfrac{x-2}{x+2} - \dfrac{x+2}{x-2} = \dfrac{2-9x}{x^2-4}.$

19. $\dfrac{y-2}{3} + \dfrac{y-3}{4} + \dfrac{y-4}{5} + \dfrac{y-5}{6} = 15.$

20. $\dfrac{1}{x - \dfrac{1}{x + \dfrac{1}{x}}} = \dfrac{x + \frac{1}{3}}{x^2}.$

21. $\dfrac{a}{x^2+x+1} + \dfrac{cx^2+c}{x^3-1} = \dfrac{c}{x-1}.$

22. $\dfrac{m+n}{x+3} = \dfrac{m-n}{x-3} + \dfrac{4m}{x^2-9}.$

23. $\dfrac{x-1}{x-2} - \dfrac{x-5}{x-6} = \dfrac{x-3}{x-4} - \dfrac{x-7}{x-8}.$

NOTE.—Subtract each member before clearing of fractions.

PROBLEMS IN SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

139. The solution of a problem in algebra consists of two distinct parts:

1. The formation of the equation;
2. The solution of the equation.

In the solution of algebraic problems, much depends upon the skill and ingenuity of the student. No general rule can be given covering all cases, but the following suggestions will be found valuable:

SUGGESTIONS.—1. *Represent the unknown quantity by one of the final letters of the alphabet.*

2. *Form an equation by indicating the operations which would be necessary to verify the result were it known.*

3. *Solve the equation thus derived.*

PROBLEMS.

1. If 16 be subtracted from six times a number, the result will equal three times the number, increased by 44; what is the number?

Let $x =$ the number.

Then $6x - 16 =$ six times the number minus 16.

And $3x + 44 =$ three times the number increased by 44.

By the conditions, $6x - 16 = 3x + 44$

$$3x = 60$$

$$x = 20,\ \textit{Ans.}$$

2. If 100 be subtracted from ten times a number, the result will equal five times the number, increased by 20; what is the number?

3. What number when doubled is as much less than 100 as the number is greater than 20?

4. What number increased by 10 and then doubled equals three times the number, minus 50?

5. If twice a number be diminished by 60 and then multiplied by 3, the result will equal twice the number, plus 24; required the number.

6. If twice a number, diminished by 12, be subtracted from three times the number, and the result be multiplied by 4, the product will equal twice the number, plus 96; what is the number?

7. What number is that to which if its $\frac{2}{7}$ be added the result will be 18?

8. What number increased by its one half, its one third, and its one fourth, will equal 500?

9. Find the number whose double exceeds its half by 210.

10. Find the number whose treble exceeds its third by 120.

11. Find the number the sum of whose half, third, and fourth exceeds the sum of its fifth, sixth, and seventh by 482.

12. What number is that whose $\frac{1}{n}$th part exceeds its $\frac{1}{m}$th part by a?

13. Divide 320 into three parts, such that three times the first part shall equal the second part, and four times the second part shall equal the third part.

14. Divide 84 into two such parts that if the first part be divided by 5 and the second by 6, the sum of the two quotients will be 16.

15. The sum of two numbers is 120, and their difference is 50; what are the numbers?

16. The sum of two numbers is 400, and their difference is 150; find the numbers.

17. The sum of two numbers is s, and their difference is d; find the numbers.

18. A horse and wagon cost $360; what was the cost of each if the wagon cost three fifths as much as the horse?

19. Find three consecutive numbers whose sum is 102.

20. Find four consecutive numbers whose sum is 98.

21. Find five consecutive numbers whose sum is 150.

22. A man gave a number of children 10 cents each, and had one dollar remaining: if he had given them 19 cents each, he would have had 1 cent remaining; how many children were there?

23. B gave some beggars 8 cents each and had $2 remaining: if he had given them 23 cents each, he would have lacked one dollar of having money enough; how much money had he?

24. M gave a cents apiece for some oranges, and had b dollars remaining: if he had given c cents apiece, he would have had d cents remaining; how many oranges did he buy?

25. A can do a piece of work in 8 days, and B in 10 days; in what time can they together do it?

Let x = the number of days.

Then $\frac{1}{x}$ = what both do in a day.

And $\frac{1}{8}$ = what A does in a day.

And $\frac{1}{10}$ = what B does in a day.

By the conditions, $\frac{1}{8} + \frac{1}{10} = \frac{1}{x}$.

Then $10x + 8x = 80$

$x = 4\frac{4}{9}$ days, *Ans.*

26. Arthur can mow a field in 10 days, Horace in 12 days, and Ross in 16 days; in what time can they together mow it?

27. A can do a piece of work in $3\frac{1}{3}$ days, B in $6\frac{1}{5}$ days, and C in 7 days; in what time can it be done by all, working together?

28. A and B can build a boat in 20 days, B and C in 24 days, and B alone in 36 days; how long will it take A, B, and C to build it, working together?

29. M and N can reap a field in a days; how long will it take each to reap it if M works twice as fast as N?

30. Two pipes fill a cistern in 20 and 30 hours respectively; how long will it take to fill the cistern if both pipes are open at the same time?

31. A tank can be filled by two pipes in 10 and 15 hours respectively, and emptied by a third pipe in 8 hours; how long will it take to fill the tank if all the pipes are left open at the same time?

32. What is the time of day when $\frac{2}{3}$ of the time past noon equals the time to midnight?

Represent the time from noon to midnight by the line *NM*, and let *T* represent the time of day.

Let $x = NT$, the time past noon.

Then $12 - x = TM$, the time to midnight.

By the conditions, $\frac{2}{3}x = 12 - x$

$$\frac{5}{3}x = 12$$

$x = 7\frac{1}{5} = 12$ minutes past seven, *Ans.*

33. What is the time of day when $\frac{1}{5}$ of the time past midnight equals the time to midnight again?

34. What is the time of day when $\frac{1}{3}$ of the time past midnight equals $\frac{5}{8}$ of the time past noon?

35. What is the time of day when $\frac{2}{3}$ of the time past 10 o'clock A. M. 2 hours ago equals $\frac{4}{9}$ of the time to midnight 2 hours hence?

36. What is the time of day when $\frac{m}{n}$th of the time past midnight equals the time to noon?

37. A is 5 years old, and B is six times as old; in how many years will B be only twice as old?

Let $x =$ the required time.

Then $5 + x =$ A's age at that time.

And $30 + x =$ B's age at that time.

By the conditions, $2(5 + x) = 30 + x$

$$10 + 2x = 30 + x$$

$x = 20$, *Ans.*

38. A's age 10 years ago was $\frac{1}{2}$ of what his age will be 10 years hence; find his age.

39. 10 years ago M was $\frac{1}{3}$ as old as N, but 4 years hence he will be $\frac{4}{5}$ as old; what is the age of each at present?

40. A's age in a years will be n times what his age was b years ago; what is his age at present?

41. E is m times as old as F, but in a years he will be n times as old; required the age of each at present.

42. M's age a years ago was $\frac{1}{n}$th of what it will be c years hence; what is his age?

43. Twelve men engaged a supper for \$20; but before paying the bill a number of them withdrew, by which each person's bill was increased \$$\frac{5}{6}$. How many withdrew?

44. Eight persons hired a coach for \$48; but before starting they took in an additional number, by which each person's bill was diminished \$2. How many were added to the party?

45. n persons engaged a boat for sailing, for which they were to pay \$$a$; but before starting a number declined going, by which each person's expenses were increased \$$b$. How many went, and how many remained?

46. At what time after 4 o'clock are the hour and minute hands of my watch together?

Let x = the distance the minute hand goes.

Then $\frac{x}{12}$ = the distance the hour hand goes.

And $x - \frac{x}{12}$ = the distance the minute hand must gain.

By the conditions, $x - \frac{x}{12} = 20$ (4 o'clock being 20 m. from 12 on the dial).

Wherefore $\frac{11x}{12} = 20$

And $x = 21\frac{9}{11}$ minutes, *Ans.*

47. At what time are the hands of a watch together—

1. Between 6 and 7 o'clock?
2. Between 8 and 9 o'clock?
3. Between 10 and 11 o'clock?

48. At what time are the hands of a watch at right angles with each other—

1. After 2 o'clock?
2. After 6 o'clock?
3. After 9 o'clock?

49. At what time are the hands of a watch opposite each other—

1. After 2 o'clock?
2. After 5 o'clock?
3. After 8 o'clock?

50. At what time are the hands of a watch 5 minutes of space apart—

1. After 3 o'clock?
2. After 6 o'clock?
3. After 10 o'clock?

51. At what time are the hands of a watch 5 minutes of time apart—

1. After 1 o'clock?
2. After 7 o'clock?
3. After 9 o'clock?

52. What time after n o'clock are the hands of a watch m minutes of space apart? When m minutes of time apart?

53. How far may a person ride in a car going at the rate of 30 miles an hour if he returns at the rate of 18 miles an hour and is gone but 10 hours?

54. A boat whose rate of sailing in still water is 6 miles an hour moves down a stream whose current is 3 miles an hour, and returns, making a round trip in 8 hours; how far did it go down stream?

55. A boat whose rate of sailing in still water is a miles an hour moves down a river whose current is b miles an hour; how far may it sail if it goes and returns in c hours?

56. A rectangle whose length is 10 feet more than its width will have its area increased 48 square feet if both dimensions are increased 2 feet; required the dimensions.

57. A rectangle is 6 feet longer and 4 feet narrower than an equivalent square; what are its dimensions?

58. A rectangle is twice as long as it is wide: if each dimension be increased 10 feet, the area will be increased 700 square feet; what are its dimensions?

59. A tree 150 feet high was broken off in a storm: the broken end remained on the stump, and the top struck 30 feet from the base; how high was the stump?

SUGGESTION.—The square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides.

60. A pole whose length was 120 feet was broken off in a storm: the one end remained on the stump, and the top struck 40 feet from the base; how much broke off?

61. A pole *a* feet high was broken off: the broken end remained on the stump, and the top struck *b* feet from the base; how high was the stump?

62. A pole standing in a stream projected 6 feet above the water: in a storm it was blown over, and was observed to disappear below the surface at a point 18 feet from the vertical position; how long was the pole? How deep was the water?

63. A pole standing in a stream projected *a* feet above the water: in a storm it was blown over, and was observed to disappear below the surface *b* feet from the vertical position; how long was the pole?

64. A general formed his men into a solid square, and had 100 men over: he increased the side of the square by 3 men, and then lacked 227 men to complete the square; how many men in the army?

65. A general formed his army consisting of 5600 men into a hollow square 8 men deep; required the number of men in the outer rank of a side.

66. A general drew up his army into a solid square, and had 96 men over: he increased the side of the square by 3 men, and lacked 525 to complete the square; how many men were in the army?

67. A invested a certain sum of money in 5% bonds bought at 85, and the same sum in 6% bonds bought at 95: he received $7 more income from the latter than from the former; what sum was invested in each?

68. B invests a certain sum of money in 4% bonds bought at 126, and twice as much in 4½% bonds bought at 130: he receives $307 more from the latter than from the former; how much was invested in each?

69. A doubles his capital every year, but at the end of each year deducts $2,500 for living expenses: at the end of four years he has eleven times his original stock; how much had he at first?

70. A boy went to a store and spent $½ more than ½ of his money; at a second store he spent $½ more than ½ of what he had remaining; and at a third store he spent $½ more than ½ of what he had yet remaining, and then found that he had $5 left; how much had he at first?

71. A man who can row 9 miles an hour in still water rows uniformly down a stream one hour; then, resting, he floats with the current half an hour, and rows back in 3 hours; how rapid is the current?

72. Two men, A and B, are a miles apart, and travel toward each other, A at the rate of m miles an hour, and B at the rate of n miles an hour; how far will each travel before they meet?

73. Two men, A and B, are a miles apart, and travel in the same direction, A at the rate of m miles an hour and B at the rate of n miles an hour; how far will each travel before they are together?

74. If a pounds of sea water contain b pounds of salt, how much salt must be added so that m pounds of the mixture may contain n pounds of salt?

75. Divide 80 into four such parts that, if the first be increased by 3, the second diminished by 3, the third multiplied by 3, and the fourth divided by 3, the results will all be equal.

76. The fore wheel of a wagon is 10 feet, and the hind wheel 12 feet, in circumference; how far must the wagon go in order that the fore wheel may make 264 revolutions more than the hind wheel?

77. The fore wheel of a carriage is m feet, and the hind wheel n feet, in circumference; how far must the carriage go in order that the fore wheel may make t revolutions more than the hind wheel?

78. A and B had equal sums of money: A bought sheep at $12 each, and had $40 remaining; B bought twice as many lambs at $8 each, and lacked $40 of having money enough to pay for them; how much did each have at first?

79. M and N had equal sums of money: M bought horses at $\$a$ each, and had $\$b$ remaining; N bought n times as many cows at $\$c$ each, and lacked $\$d$ of having money enough to pay for them; how much money had each?

80. A receives $3 a day for his labor, on condition that he forfeit $1 each day he is idle: at the expiration of 50 days he receives $110; how many days was he idle?

81. A receives $\$a$ a day for his labor, on condition that he forfeit $\$b$ each day he is idle: at the expiration of n days he receives $\$c$; required the number of idle days.

82. Two men, M and N, 36 miles apart, set out, M 45 minutes before N, and travel in the same direction: M travels 4 miles an hour, and N 7 miles an hour; how far will each have traveled when N overtakes M?

83. A man starts from a place and travels 8 miles an hour: 10 hours after he is followed by another, who travels 12 miles an hour; in what time will the second overtake the first?

84. E starts from a place and travels 25 miles a day: 12 days after he is followed by F, who travels 35 miles a day: how far will F travel to overtake E?

85. A courier starts from a place and travels a miles a day: n days after he is followed by another, who travels b miles a day; in what time will the second overtake the first?

86. A grocer buys two kinds of sugar, one at 8 cents a pound and the other at 11 cents a pound; how much of each kind must he take to make a mixture of 60 pounds worth 10 cents a pound?

87. A man buys two kinds of tea, one at 60 cents a pound and the other at 90 cents a pound; how much of each kind must he take to make a mixture of 120 pounds worth 80 cents a pound?

88. A merchant buys two kinds of coffee, one at a cents a pound and the other at b cents a pound; how much of each kind must he take to make a mixture of m pounds worth c cents a pound?

89. The ten's digit of a number exceeds the unit's digit by 2; and if the number increased by 3 be divided by the sum of its digits, the quotient will be 7; required the number.

90. In a number of two places the ten's digit is 6 more than the unit's digit; and if the number increased by 15 be divided by the difference of its digits, the quotient will be 18; what is the number?

91. A farmer has a square field of land which is worth, at $120 an acre, nine times as much as it would cost to inclose it by a fence at one dollar a rod; required the length of one side.

92. What is the length of one side of a square field which contains as many acres as there are rods in its perimeter?

93. What is the diameter of a circular field which contains as many acres as there are rods in its circumference?

94. A man loans $28,080, part at 4% and part at 5%; how much does he loan at each rate if he receives equal sums of interest for each part?

95. A man has $49.20, consisting of dollars, dimes, and cents: he has twice as many dimes as dollars, and three times as many cents as dollars; how many pieces of each kind does he have?

96. A man has a certain sum of money, consisting of dollars, quarters, and dimes: he has twice as many quarters as dimes, and twice as many dollars as quarters; if he had twice as many quarters as he has dollars, and twice as many dimes as he would then have quarters, he would have $36 more. How many pieces are there of each kind?

97. A broker has two kinds of money, dimes and quarters; if he receives $5.90 for 35 pieces of money, how many of each kind must he give?

98. A broker has two kinds of money: it takes n pieces of the first kind to make a dollar, and m pieces of the second kind; how many pieces of each kind must he take, so that p pieces will make a dollar?

99. A fox is 60 leaps ahead of a hound, and takes 6 leaps while the hound takes 5; but 8 leaps of the hound are equal to 10 leaps of the fox. How many leaps will each take before they are together?

100. A general, arranging his men in a solid square, finds he has 82 men over: but in attempting to add one line of men around the entire square, he finds he wants 82 men to fill up the square; required the number of men in the army.

101. The distance from Harrisburg to Philadelphia is 105 miles: the Atlantic express leaves Harrisburg at 10.35 A. M., and arrives at Philadelphia at 1.35 P. M.; the fast line leaves Philadelphia at 11.40 A. M., and arrives at Harrisburg at 3.40 P. M.; how far from Philadelphia do they meet, if their rates of running are uniform?

102. If the length of a rectangle is increased 10 rd. and its width 5 rd., the area will be increased 500 sq. rd.; but if the length be diminished 5 rd. and the width 10 rd., the area will be diminished 550 sq. rd. Find the dimensions of the field.

SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

140. If we have the simple equation $x+y=4$, and

let $x = 1$; then $y = 3$,
or $x = 2$; then $y = 2$,
or $x = 3$; then $y = 1$,
or $x = 4$; then $y = 0$,
etc. etc.

Since any of these pairs of values will satisfy the equation, it is impossible to determine definitely the values of x and y.

But if we have the two simple equations, $x + y = 4$ and $x - y = 2$, and are required to find a pair of values which will satisfy both of these equations at the same time, we can find by inspection that

$$x = 3 \text{ and } y = 1$$

will satisfy both equations. Furthermore, this is the only pair of values which can be found that will satisfy both at the same time.

It is therefore evident that we must have two equations when we use two unknown quantities in order to determine their values.

141. Simultaneous Equations are those which are satisfied by the same values of the unknown quantities.

Thus, if $x = 3$ and $y = 4$, then
$x + y = 7$ and $2x - y = 2$

are simultaneous equations.

142. Independent Equations are those which cannot be made to assume the same form.

Thus, $x + y = 7$ and $2x - y = 2$ are independent equations.

But $x + y = 7$ and $2x + 2y = 14$ are not independent, since either equation can be derived from the other.

ELIMINATION.

143. Elimination is the process of deducing from two or more simultaneous equations a less number of equations containing a less number of unknown quantities.

There are three principal methods of elimination:

1. By comparison;
2. By substitution;
3. By addition or subtraction.

CASE I.

144. Elimination by comparison.

Elimination by Comparison is the process of eliminating by finding an expression for the same unknown quantity from each equation, and placing them equal to each other.

1. Solve the equations $\begin{cases} 3x + 7y = 134. & (1) \\ 4x - 2y = 20. & (2) \end{cases}$

From (1), $$x = \frac{134 - 7y}{3} \qquad (3)$$

From (2), $$x = \frac{20 + 2y}{4} \qquad (4)$$

From (3) and (4), $$\frac{134 - 7y}{3} = \frac{20 + 2y}{4} \text{ (Ax. 9)} \qquad (5)$$

Clearing of fractions, $$536 - 28y = 60 + 6y \qquad (6)$$

Or, $$-34y = -476 \qquad (7)$$

$$y = 14$$

Substituting the value of y in (3),

$$x = \frac{134 - 98}{3} = 12.$$

Ans., $x = 12$, $y = 14$.

Therefore, to eliminate by comparison,

RULE.

Find an expression for the same unknown quantity from each equation, place these values equal to each other, and reduce the resulting equation.

EXAMPLES.

Solve the following equations by comparison:

2. $\begin{cases} 3x - y = 2. \\ x + y = 6. \end{cases}$

3. $\begin{cases} 2x + 3y = 13. \\ 3x + 2y = 12. \end{cases}$

4. $\begin{cases} 3x - y = 2. \\ 10x + 2y = 60. \end{cases}$

5. $\begin{cases} x + y = 9. \\ 2x + 3y = 20. \end{cases}$

6. $\begin{cases} 2x + 5y = 18. \\ 3x + 4y = 20. \end{cases}$

7. $\begin{cases} 7x + 3y = 58. \\ 3x - 5y = 6. \end{cases}$

8. $\begin{cases} 11x - 6y = 4. \\ 4x - y = 5. \end{cases}$

9. $\begin{cases} 4x + 3y = 41. \\ 4x - 2y = 6. \end{cases}$

10. $\begin{cases} 2x + 3y = 24. \\ 10x + y = 50. \end{cases}$

11. $\begin{cases} 17x + 11y = 45. \\ 9x - 7y = 11. \end{cases}$

12. $\begin{cases} x + y = 20. \\ 4x - 5y = 8. \end{cases}$

13. $\begin{cases} x - y = 4. \\ 3x + 5y = 44. \end{cases}$

14. $\begin{cases} 4x - y = 6. \\ 17x + 10y = 168. \end{cases}$

15. $\begin{cases} 5x + 3y = 26. \\ 11x - 10y = 24. \end{cases}$

16. $\begin{cases} \frac{2}{3}x + \frac{1}{2}y = 9. \\ \frac{5}{6}x - \frac{3}{2}y = 7. \end{cases}$

17. $\begin{cases} \frac{1}{2}x + \frac{2}{3}y = 8. \\ \frac{3}{4}x + \frac{7}{9}y = 10. \end{cases}$

CASE II.

145. Elimination by substitution.

Elimination by substitution is the process of eliminating by finding an expression for the value of an unknown quantity from one equation, and substituting it in the other.

1. Solve the equations $\begin{cases} 3x + 2y = 13. & (1) \\ 2x + 3y = 12. & (2) \end{cases}$

From (1), $$x = \frac{13 - 2y}{3} \qquad (3)$$

Substituting this value in (2), $$2\left(\frac{13 - 2y}{3}\right) + 3y = 12 \qquad (4)$$

Clearing of fractions, $$26 - 4y + 9y = 36 \qquad (5)$$

Or, $$5y = 10$$

$$y = 2$$

Substituting this value of y in (3), $$x = \frac{13 - 4}{3} = 3$$

Ans., $x = 3$, $y = 2$.

Therefore, to eliminate by substitution,

RULE.

Find an expression for one of the unknown quantities from either equation, substitute this value for the same unknown quantity in the other equation, and reduce the resulting equation.

EXAMPLES.

Solve the following equations by substitution:

2. $\begin{cases} x + 4y = 14. \\ 6x + 2y = 18. \end{cases}$

3. $\begin{cases} x - y = 1. \\ 5x + 7y = 29. \end{cases}$

4. $\begin{cases} x - y = 8. \\ 2x + 3y = 61. \end{cases}$

5. $\begin{cases} x + 2y = 14. \\ 3x + 7y = 47. \end{cases}$

6. $\begin{cases} 4x + 3y = 36. \\ 5x + y = 23. \end{cases}$

7. $\begin{cases} 5x - 3y = 17. \\ 4x - 2y = 16. \end{cases}$

8. $\begin{cases} 6x - 3y = 2. \\ 2x - 3y = 0. \end{cases}$

9. $\begin{cases} 12x - 5y = 46. \\ 3x - 2y = 4. \end{cases}$

10. $\begin{cases} 4x + 11y = 44. \\ 6x - 8y = 66. \end{cases}$

11. $\begin{cases} 4x + 2y = 22. \\ 6x - 3y = 3. \end{cases}$

12. $\begin{cases} 5x + 2y = 20. \\ 3x + 2y = 16. \end{cases}$

13. $\begin{cases} 6x + 5y = 1. \\ 3y - 2x = 23. \end{cases}$

14. $\begin{cases} 11x + 8y = -3. \\ 9y - 4x = 13. \end{cases}$

15. $\begin{cases} x + y = -11. \\ 7x - 2y = -14. \end{cases}$

CASE III.

146. Elimination by addition or subtraction.

Elimination by addition or subtraction is the process of eliminating by adding or subtracting the equations when the coefficients are alike or are made alike.

1. Solve the equations $\begin{cases} 2x + 3y = 12. & (1) \\ 3x + 2y = 13. & (2) \end{cases}$

Multiplying (1) by 3, $6x + 9y = 36$ (3)

Multiplying (2) by 2, $6x + 4y = 26$ (4)

Subtracting (4) from (3), $5y = 10$

$y = 2$

Substituting this value of y in (1), $2x + 6 = 12$

$2x = 6$

$x = 3$

Ans., $x = 3$, $y = 2$.

2. Solve the equations $\begin{cases} 6x + 2y = 28. & (1) \\ 7x - 3y = 6. & (2) \end{cases}$

Multiplying (1) by 3, $18x + 6y = 84$ (3)

Multiplying (2) by 2, $14x - 6y = 12$ (4)

Adding these equations, $32x = 96$ (5)

$x = 3$

Substituting this value in (1), $18 + 2y = 28$

$2y = 10$

$y = 5$

Ans., $x = 3$, $y = 5$.

Therefore, to eliminate by addition or subtraction,

RULE.

If necessary, multiply or divide the given equations by such a number as will make the coefficients of one of the unknown quantities equal. When the signs of the equal coefficients are unlike, add the equations; when they are alike, subtract them. Solve the resulting equation.

EXAMPLES.

Solve the following equations by addition or subtraction:

3. $\begin{cases} x + y = 6. \\ x - y = 4. \end{cases}$

4. $\begin{cases} 2x + 3y = 22. \\ 2x - y = 6. \end{cases}$

5. $\begin{cases} 11x - 2y = 5. \\ 10x + 2y = 16. \end{cases}$

6. $\begin{cases} 8x - 7y = 0. \\ 4x + 3y = 52. \end{cases}$

7. $\begin{cases} 3x + y = 6. \\ 5x + 2y = 11. \end{cases}$

8. $\begin{cases} 9x - 6y = 3. \\ 4x - y = 18. \end{cases}$

9. $\begin{cases} 8x + 3y = 5. \\ 12x + 15y = 11. \end{cases}$

10. $\begin{cases} 3x - \frac{1}{2}y = 17. \\ 2x - y = 6. \end{cases}$

11. $\begin{cases} 6x + 5y = 65. \\ 10x - 3y = 29. \end{cases}$

12. $\begin{cases} 8x - 2y = 16. \\ 5x + 7y = 43. \end{cases}$

13. $\begin{cases} 5x - 11y = -117. \\ 7x + 6y = 93. \end{cases}$

14. $\begin{cases} 5x + 4y = -17. \\ 6x - 3y = -36. \end{cases}$

15. $\begin{cases} 19x + 13y = 45. \\ 14x - 2y = 10. \end{cases}$

16. $\begin{cases} 18x - 20y = 44. \\ 17x - 15y = 26. \end{cases}$

17. $\begin{cases} \frac{2}{3}x + \frac{3}{4}y = 10. \\ \frac{1}{2}x - \frac{1}{4}y = 1. \end{cases}$

18. $\begin{cases} 10x - y = 9. \\ 10y - x = 108. \end{cases}$

MISCELLANEOUS EXAMPLES.

147. Reduce the following equations to their simplest form, and apply the method best adapted to each example:

1. Solve the equations
$$\begin{cases} \dfrac{\dfrac{x+y}{5} - \dfrac{x-y}{2}}{\dfrac{2x+y}{4} + \dfrac{3x-4y}{2}} = \dfrac{1}{5}. & (1) \\[2ex] x - y = 2. & (2) \end{cases}$$

From (1), $\quad x + y - \frac{5x - 5y}{2} = \frac{2x + y}{4} + \frac{3x - 4y}{2} \qquad (3)$

Clearing (3) of fractions,

$4x + 4y - 10x + 10y = 2x + y + 6x - 8y \qquad (4)$

Collecting terms, $\quad 21y - 14x = 0 \qquad (5)$

Or, $\quad 3y - 2x = 0 \qquad (6)$

Multiplying (2) by 3, $\quad -3y + 3x = 6 \qquad (7)$

Adding (6) and (7), $\quad x = 6$

Substituting this value of y in (2), $y = 4$.

Ans., $x = 6, y = 4$.

Solve the following:

2. $$\begin{cases} \dfrac{5x-6y}{6}+\dfrac{5y}{3}=13. \\[2ex] \dfrac{5x}{6}-\dfrac{7x-4y}{3}=7. \end{cases}$$

3. $$\begin{cases} \dfrac{x}{2}+\dfrac{3y}{2}=11. \\[2ex] \dfrac{y+x}{2}+\dfrac{x-y}{2}=2. \end{cases}$$

4. $$\begin{cases} \dfrac{x+3}{2}+\dfrac{y+4}{3}=6. \\[2ex] \dfrac{2x-5}{3}-\dfrac{4y-18}{6}=0. \end{cases}$$

5. $$\begin{cases} \dfrac{x+4}{3}-\dfrac{26-y}{8}=3. \\[2ex] \dfrac{x-1}{3}+\dfrac{5(y-2)}{4}=13. \end{cases}$$

6. $$\begin{cases} \dfrac{x+2}{y+3}=\dfrac{x-2}{y-3}. \\[2ex] \dfrac{3x-5}{2y-2}=\dfrac{6x-14}{4y-2}. \end{cases}$$

7. $$\begin{cases} \dfrac{x+8}{3}+4y=28. \\[2ex] \dfrac{x+y}{2}=27-(3y+x). \end{cases}$$

8. $$\begin{cases} \dfrac{x+y}{8}+\dfrac{x-y}{6}=10. \\[2ex] \dfrac{x+y}{4}-\dfrac{x-y}{3}=10. \end{cases}$$

9. $$\begin{cases} \dfrac{x+1}{y-2}=\dfrac{x+3}{y-4}. \\[2ex] \dfrac{x-5}{y+6}=\dfrac{x-7}{y-9}. \end{cases}$$

10. $$\begin{cases} \dfrac{x+y}{x-y}=\dfrac{7}{8}. \\[2ex] \dfrac{2x-6}{3x-5}=\dfrac{4}{5}. \end{cases}$$

11. $$\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=m. \\[2ex] \dfrac{x}{b}+\dfrac{y}{a}=n. \end{cases}$$

12. $$\begin{cases} \dfrac{3x-2}{7}-\dfrac{4x-2y-1}{5}=1. \\[2ex] \dfrac{x+2}{y-2}=\dfrac{x+5}{y+6}. \end{cases}$$

13. $$\begin{cases} \dfrac{y-2}{5}-\dfrac{10-y}{3}=\dfrac{x-10}{4}. \\[2ex] \dfrac{2x+4}{3}-\dfrac{2y+x}{8}=\dfrac{y+13}{4}. \end{cases}$$

14. $$\begin{cases} x(y+2)+4=y(x+3)-6. \\ 3(x+y)-y=4(x-y)+32. \end{cases}$$

15. $$\begin{cases} (x+2)(y-2)=19+(x-4)(y-3). \\ 3(x-2)+4(y+2)=51. \end{cases}$$

16. $$\begin{cases} \dfrac{x+2}{4}=\dfrac{y+3}{6}=\dfrac{x+y}{5}. \end{cases}$$

17. $$\begin{cases} \dfrac{x+4}{2}+\dfrac{y-2}{3}=\dfrac{x+6}{2}+1. \\[2ex] \dfrac{3x-2}{5}-\dfrac{4y-6}{13}=y-x. \end{cases}$$

18. $$\begin{cases} \dfrac{10}{3x+5y+4}=\dfrac{12}{4x+8y+2}. \\[2ex] \dfrac{x-2}{3}=\dfrac{y-3}{4}. \end{cases}$$

19. $$\begin{cases} \dfrac{2x+y}{x-3y}=9. \\[2ex] \dfrac{x+4}{3}-\dfrac{x}{2}-\dfrac{y-1}{4}+\dfrac{7}{4}=\dfrac{3}{20}. \end{cases}$$

20. $$\begin{cases} \dfrac{\dfrac{3x+2y}{4}}{\dfrac{4x-2y}{3}}-2=7-\dfrac{\dfrac{5x-2y}{2}}{\dfrac{8x-4y}{3}}. \\[3ex] 38y=27x+218. \end{cases}$$

21. $$\begin{cases} \dfrac{4}{3x+4y-2}=\dfrac{3}{3y+4x-5}. \\[2ex] \dfrac{5}{6x+7y+2}=\dfrac{3}{13x-4y+7}. \end{cases}$$

22. $$\begin{cases} 2x+3y-6=\dfrac{2(y-x)}{3}+5\tfrac{1}{3}+\dfrac{x+y}{5}. \\[2ex] (x+2)(y-2)=(x-1)(y+1). \end{cases}$$

23. $$\begin{cases} \dfrac{2x-y}{4}+4=\dfrac{3y-x}{7}+\dfrac{5}{2}. \\ \dfrac{4x+y}{11}+1=\dfrac{4y+x}{7}+x-5. \end{cases}$$

24. $$\begin{cases} \dfrac{x+y}{9}+\dfrac{x-y}{2}=1. \\ \dfrac{x+y}{6}-\dfrac{x-y}{5}=3\frac{2}{5}. \end{cases}$$

28. $$\begin{cases} .3x+.5y=1.45. \\ .09x+.7y=4.01. \end{cases}$$

25. $$\begin{cases} \dfrac{x+y}{x-y}=\dfrac{3}{4}. \\ \dfrac{x+y+2}{x-y-2}=\dfrac{3}{7}. \end{cases}$$

29. $$\begin{cases} .16x-.02y=.52. \\ .11x-.003y=.422. \end{cases}$$

26. $$\begin{cases} \dfrac{7x+3}{5+x-y}=\dfrac{4}{7}. \\ x+y-10=0. \end{cases}$$

30. $$\begin{cases} .5x+3y=40. \\ 8x+.33\frac{1}{3}y=68. \end{cases}$$

27. $$\begin{cases} \dfrac{3x+4}{5x-2}=\dfrac{4-3y}{6-5y}. \\ \dfrac{x}{2}+\dfrac{y+5}{3}=3\frac{1}{2}. \end{cases}$$

31. $$\begin{cases} \dfrac{.001x+.01y}{2}=2.5. \\ \dfrac{.01x+.001y}{7}=2.9. \end{cases}$$

148. Literal Simultaneous Equations.—It will generally be found most convenient in solving literal simultaneous equations to eliminate by addition or subtraction.

1. Solve the equations $\begin{cases} ax+by=c. & (1) \\ mx+ny=d. & (2) \end{cases}$

Multiplying (1) by m, $amx + bmy = cm$ (3)

Multiplying (2) by a, $amx + any = ad$ (4)

Subtracting (3) from (4), $any - bmy = ad - cm$ (5)

Whence, $y = \dfrac{ad - cm}{an - bm}$, *Ans.*

Multiplying (1) by n, $anx + bny = cn$ (6)

Multiplying (2) by b, $bmx + bny = bd$ (7)

Subtracting (7) from (6), $anx - bmx = cn - bd$ (8)

Whence, $x = \dfrac{cn - bd}{an - bm}$, *Ans.*

14. $$\begin{cases} x(y+2)+4=y(x+3)-6. \\ 3(x+y)-y=4(x-y)+32. \end{cases}$$

15. $$\begin{cases} (x+2)(y-2)=19+(x-4)(y-3). \\ 3(x-2)+4(y+2)=51. \end{cases}$$

16. $$\left\{ \frac{x+2}{4}=\frac{y+3}{6}=\frac{x+y}{5}. \right.$$

17. $$\begin{cases} \dfrac{x+4}{2}+\dfrac{y-2}{3}=\dfrac{x+6}{2}+1. \\[2ex] \dfrac{3x-2}{5}-\dfrac{4y-6}{13}=y-x. \end{cases}$$

18. $$\begin{cases} \dfrac{10}{3x+5y+4}=\dfrac{12}{4x+8y+2}. \\[2ex] \dfrac{x-2}{3}=\dfrac{y-3}{4}. \end{cases}$$

19. $$\begin{cases} \dfrac{2x+y}{x-3y}=9. \\[2ex] \dfrac{x+4}{3}-\dfrac{x}{2}-\dfrac{y-1}{4}+\dfrac{7}{4}=\dfrac{3}{20}. \end{cases}$$

20. $$\begin{cases} \dfrac{\dfrac{3x+2y}{4}}{\dfrac{4x-2y}{3}}-2=7-\dfrac{\dfrac{5x-2y}{2}}{\dfrac{8x-4y}{3}}. \\[3ex] 38y=27x+218. \end{cases}$$

21. $$\begin{cases} \dfrac{4}{3x+4y-2}=\dfrac{3}{3y+4x-5}. \\[2ex] \dfrac{5}{6x+7y+2}=\dfrac{3}{13x-4y+7}. \end{cases}$$

22. $$\begin{cases} 2x+3y-6=\dfrac{2(y-x)}{3}+5\tfrac{1}{3}+\dfrac{x+y}{5}. \\[2ex] (x+2)(y-2)=(x-1)(y+1). \end{cases}$$

23. $\begin{cases} \dfrac{2x-y}{4}+4=\dfrac{3y-x}{7}+\dfrac{5}{2}. \\ \dfrac{4x+y}{11}+1=\dfrac{4y+x}{7}+x-5. \end{cases}$

24. $\begin{cases} \dfrac{x+y}{9}+\dfrac{x-y}{2}=1. \\ \dfrac{x+y}{6}-\dfrac{x-y}{5}=3\frac{2}{5}. \end{cases}$

25. $\begin{cases} \dfrac{x+y}{x-y}=\dfrac{3}{4}. \\ \dfrac{x+y+2}{x-y-2}=\dfrac{3}{7}. \end{cases}$

26. $\begin{cases} \dfrac{7x+3}{5+x-y}=\dfrac{4}{7}. \\ x+y-10=0. \end{cases}$

27. $\begin{cases} \dfrac{3x+4}{5x-2}=\dfrac{4-3y}{6-5y}. \\ \dfrac{x}{2}+\dfrac{y+5}{3}=3\frac{1}{2}. \end{cases}$

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148. Literal Simultaneous Equations.—It will generally be found most convenient in solving literal simultaneous equations to eliminate by addition or subtraction.

1. Solve the equations $\begin{cases} ax+by=c. & (1) \\ mx+ny=d. & (2) \end{cases}$

Multiplying (1) by m, $amx+bmy=cm$ (3)

Multiplying (2) by a, $amx+any=ad$ (4)

Subtracting (3) from (4), $any-bmy=ad-cm$ (5)

Whence, $y=\dfrac{ad-cm}{an-bm}$, *Ans.*

Multiplying (1) by n, $anx+bny=cn$ (6)

Multiplying (2) by b, $bmx+bny=bd$ (7)

Subtracting (7) from (6), $anx-bmx=cn-bd$ (8)

Whence, $x=\dfrac{cn-bd}{an-bm}$, *Ans.*

EXAMPLES.

Solve the following:

2. $\begin{cases} x + y = a. \\ x - y = b. \end{cases}$

3. $\begin{cases} x + y = m - n. \\ x - y = m + n. \end{cases}$

4. $\begin{cases} x + y = 2m. \\ x - y = 2n. \end{cases}$

5. $\begin{cases} ax + by = c. \\ bx + ay = d. \end{cases}$

6. $\begin{cases} 2x + 3y = a. \\ 3x + 2y = b. \end{cases}$

7. $\begin{cases} \dfrac{x}{m} - \dfrac{y}{n} = a. \\ \dfrac{x}{n} + \dfrac{y}{m} = b. \end{cases}$

8. $\begin{cases} mx + ny = 0. \\ x - y = a. \end{cases}$

9. $\begin{cases} \dfrac{x}{m} + \dfrac{y}{n} = 2mn. \\ x + y = mn(m + n). \end{cases}$

10. $\begin{cases} \dfrac{x}{m} - \dfrac{y}{n} = 1. \\ \dfrac{x}{n} + \dfrac{y}{m} = 1. \end{cases}$

11. $\begin{cases} \dfrac{x}{m + n} + \dfrac{y}{m - n} = 1. \\ x - y = m. \end{cases}$

12. $\begin{cases} x + y = a + b. \\ x - y = a - b. \end{cases}$

13. $\begin{cases} \dfrac{ax}{2} + \dfrac{by}{3} = 2ab. \\ \dfrac{ax}{3} + \dfrac{by}{2} = 3ab. \end{cases}$

14. $\begin{cases} \dfrac{x + y}{m} + \dfrac{x - y}{n} = 1. \\ x + y = m. \end{cases}$

15. $\begin{cases} (a + b)x - cy = bc. \\ x + y = a + b. \end{cases}$

16. $\begin{cases} \dfrac{ax + by}{2} = m. \\ \dfrac{ax - by}{2} = n. \end{cases}$

17. $\begin{cases} \dfrac{x}{2} + \dfrac{y}{4} = a. \\ \dfrac{x}{7} - \dfrac{y}{9} = b. \end{cases}$

18. $\begin{cases} .5x + .7y = m. \\ .05x - .9y = n. \end{cases}$

19. $\begin{cases} ax + by = \dfrac{a^2 + b^2}{ab^2}. \\ bx - ay = \dfrac{a^2 - b^2}{a^2b}. \end{cases}$

20. $$\begin{cases} \dfrac{x}{a+b} - \dfrac{y}{a-b} = \dfrac{1}{a-b}. \\[2ex] \dfrac{x}{a-b} + \dfrac{y}{a+b} = \dfrac{1}{a+b}. \end{cases}$$

21. $$\begin{cases} \dfrac{x+y}{a+b} + \dfrac{x-y}{a-b} = \dfrac{2}{a-b}. \\[2ex] \dfrac{x}{a} + \dfrac{y}{b} = \dfrac{a-b}{ab}. \end{cases}$$

22. $$\begin{cases} \dfrac{x+y}{m} + \dfrac{x-y}{n} = 4. \\[2ex] \dfrac{x}{m+n} + \dfrac{y}{m-n} = 2. \end{cases}$$

23. $$\begin{cases} \dfrac{x}{n} + \dfrac{y}{m} = p. \\[2ex] ax + by = c. \end{cases}$$

149. When the unknown quantities occur in the denominators, it is better, generally, not to clear of fractions.

1. Solve the equations $$\begin{cases} \dfrac{2}{x} + \dfrac{3}{y} = 4. & (1) \\[2ex] \dfrac{3}{x} + \dfrac{2}{y} = 1. & (2) \end{cases}$$

Multiplying (1) by 3, $\frac{6}{x} + \frac{9}{y} = 12$ (3)

Multiplying (2) by 2, $\frac{6}{x} + \frac{4}{y} = 2$ (4)

Subtracting (4) from (3), $\frac{5}{y} = 10$

Whence, $y = \frac{1}{2}$, *Ans.*

Substituting this value of y in (1), $\frac{2}{x} + 6 = 4$

Whence, $x = -1$, Ans.

Solve the following equations:

2. $\frac{[illegible]}{x} - \frac{[illegible]}{y} = 1$; $\frac{[illegible]}{x} - \frac{[illegible]}{y} = 1$.

3. $\frac{[illegible]}{x} - \frac{[illegible]}{y} = 14$; $\frac{[illegible]}{x} - \frac{[illegible]}{y} = 2$.

4. $\frac{[illegible]}{x} - \frac{[illegible]}{y} = 32$; $\frac{[illegible]}{x} - \frac{[illegible]}{y} = 9$.

5. $\frac{[illegible]}{x} \; \frac{[illegible]}{y} = 19$; $\frac{3}{x} - \frac{4}{y} = 24$.

6. $\frac{m}{x} - \frac{n}{y} = 1$; $\frac{n}{x} - \frac{m}{y} = 1$.

7. $\frac{[illegible]}{x} - \frac{[illegible]}{[illegible]} = [illegible]$; $\frac{[illegible]}{[illegible]} - \frac{[illegible]}{[illegible]} = [illegible]$.

8. $\frac{[illegible]}{x} - \frac{[illegible]}{y} = \frac{[illegible]}{[illegible]}$; $\frac{4}{x} - \frac{5}{y} = \frac{[illegible]}{[illegible]}$.

9. $\frac{[illegible]}{x} - \frac{[illegible]}{y} = m$; $\frac{3}{x} - \frac{2}{y} = n$.

10. $\frac{a}{x} - \frac{b}{y} = 4$; $\frac{b}{x} - \frac{a}{y} = 5$.

11. $\frac{a}{x} - \frac{b}{y} = m$; $\frac{b}{x} - \frac{a}{y} = n$.

12. $\begin{cases} 3(x - 3y) = 2(3x - y) - 38. \\ 5(x - y) = 3(6y - 5x) - 36. \end{cases}$

13. $\begin{cases} 2x - y = 3. \\ \frac{1}{5}(2x + 5y) + 1 = \frac{2}{3}(5x - y - 3). \end{cases}$

14. $\begin{cases} \frac{1}{7}(x + 2) + \frac{1}{4}(2y - x) = 3x - 32. \\ \frac{1}{3}(2x + y + 2) - \frac{1}{5}(2y + x - 2) = 3y - 24. \end{cases}$

PROBLEMS IN SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

150.—1. A bought 5 calves and 6 sheep for \$89, and at the same price 7 calves and 8 sheep for \$121; what was the price of each per head?

Let x = the price of a calf,
And y = the price of a sheep.
Then $5x + 6y = 89$ (1)
And $7x + 8y = 121$ (2)
Multiplying (1) by 7, $35x + 42y = 623$ (3)
Multiplying (2) by 5, $35x + 40y = 605$ (4)
Subtracting (4) from (3), $2y = 18$
$y = 9$
Substituting this value of y in (1), $5x + 54 = 89$
$5x = 35$
$x = 7$
Ans., calves, \$7; sheep, \$9.

2. If 6 lbs. of sugar and 10 lbs. of tea cost \$6.30, and at the same price 10 lbs. of sugar and 6 lbs. of tea cost \$4.10, what is the price of each per pound?

3. A hired 10 men and 8 boys for \$30 a day, and at the same rate 7 men and 12 boys for \$29 a day; how much did each earn in a day?

4. A drover sold 8 horses and 14 cows for \$1,060, and at the same rate 12 horses and 20 cows for \$1,560; what was the price of each per head?

5. A mixed sugar worth 6 cents and 9 cents a pound so as to form a mixture of 60 pounds worth 7 cents a pound; how many pounds of each kind did he take?

6. A boy bought 12 oranges and 10 bananas for 56 cents; but the price having fallen $\frac{1}{2}$, he sold 8 oranges and 3 bananas for 15 cents; required the first cost of each.

7. A bought a oranges and b apples for n cents, and at the same rate b oranges and a apples for m cents; required the price of each.

Solve the following equations:

2. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = a. \\[2ex] \dfrac{1}{x} - \dfrac{1}{y} = b. \end{cases}$

3. $\begin{cases} \dfrac{2}{x} + \dfrac{5}{y} = 14. \\[2ex] \dfrac{3}{x} - \dfrac{2}{y} = 2. \end{cases}$

4. $\begin{cases} \dfrac{5}{x} + \dfrac{6}{y} = 33. \\[2ex] \dfrac{8}{x} - \dfrac{5}{y} = 9. \end{cases}$

5. $\begin{cases} \dfrac{7}{x} - \dfrac{3}{y} = 19. \\[2ex] \dfrac{3}{x} + \dfrac{4}{y} = 24. \end{cases}$

6. $\begin{cases} \dfrac{m}{x} + \dfrac{n}{y} = 1. \\[2ex] \dfrac{n}{x} + \dfrac{m}{y} = 1. \end{cases}$

7. $\begin{cases} \dfrac{4}{x} + \dfrac{5}{2y} = 12. \\[2ex] \dfrac{3}{2x} + \dfrac{5}{3y} = 18. \end{cases}$

8. $\begin{cases} \dfrac{9}{x} - \dfrac{5}{y} = \dfrac{2}{3}. \\[2ex] \dfrac{4}{x} + \dfrac{5}{y} = \dfrac{3}{2}. \end{cases}$

9. $\begin{cases} \dfrac{2}{x} + \dfrac{3}{y} = m. \\[2ex] \dfrac{3}{x} - \dfrac{2}{y} = n. \end{cases}$

10. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = 4. \\[2ex] \dfrac{b}{x} + \dfrac{a}{y} = 5. \end{cases}$

11. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = m. \\[2ex] \dfrac{b}{x} + \dfrac{a}{y} = n. \end{cases}$

12. $\begin{cases} 3(x + 3y) = 2(3x - y) + 38. \\ 5x - y = 3(6y - 5x) - 36. \end{cases}$

13. $\begin{cases} 2x = y + 3. \\[1ex] \dfrac{1}{5}(2x + 5y) + 1 = \dfrac{2}{3}(5x - y - 3). \end{cases}$

14. $\begin{cases} \dfrac{1}{7}(x + 2) + \dfrac{1}{4}(2y - x) = 3x - 32. \\[2ex] \dfrac{1}{3}(2x + y + 2) - \dfrac{1}{5}(2y + x - 2) = 3y - 24. \end{cases}$

PROBLEMS IN SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

150.—1. A bought 5 calves and 6 sheep for $89, and at the same price 7 calves and 8 sheep for $121; what was the price of each per head?

Let $x =$ the price of a calf,

And $y =$ the price of a sheep.

Then $5x + 6y = 89$ (1)

And $7x + 8y = 121$ (2)

Multiplying (1) by 7, $35x + 42y = 623$ (3)

Multiplying (2) by 5, $35x + 40y = 605$ (4)

Subtracting (4) from (3), $2y = 18$

$y = 9$

Substituting this value of y in (1), $5x + 54 = 89$

$5x = 35$

$x = 7$

Ans., calves, $7; sheep, $9.

2. If 6 lbs. of sugar and 10 lbs. of tea cost $6.30, and at the same price 10 lbs. of sugar and 6 lbs. of tea cost $4.10, what is the price of each per pound?

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6. A boy bought 12 oranges and 10 bananas for 56 cents; but the price having fallen $\frac{1}{2}$, he sold 8 oranges and 3 bananas for 15 cents; required the first cost of each.

7. A bought a oranges and b apples for n cents, and at the same rate b oranges and a apples for m cents; required the price of each.

8. If the length of a rectangle be increased [illegible] ft. and the width [illegible] ft., the area will be increased [illegible] sq. ft.; but if the length be diminished [illegible] ft. and the width [illegible] ft., the area will be diminished [illegible] sq. ft. Required the dimensions.

Suggestion.—Let x = the length and y = the width; then $\overline{x-[illegible]}\cdot\overline{y-[illegible]} = xy - [illegible]$, and $\overline{x-[illegible]}\cdot\overline{y-[illegible]} = xy - [illegible]$.

9. If the length of a rectangle is increased [illegible] rd. and its width [illegible] rd., the area will be increased one acre; but if the length be diminished [illegible] rd. and the width [illegible] rd., the area will be diminished [illegible] acre: find the dimensions of the field.

10. If the length of a rectangular field is increased 10 rd., the area will be increased [illegible] acre; but if the width be increased [illegible] rd., the area will be increased [illegible] acres: find the dimensions of the field.

11. There is a number consisting of two digits: the sum of the digits is 11, and if [illegible] be subtracted from the number, the digits will be inverted: find the number.

12. Find a number of two digits which, being increased by the number expressed by the digits inverted, equals 77, and diminished by the same number, equals 9.

13. If 18 be subtracted from a number of two places, the digits will be inverted; but if 18 be added to the number, the sum will be 21 less than twice the number when the digits are inverted: what is the number?

14. A and B play marbles: A loses to B as many as B has, then B loses to A as many as A now has, after which it was found that each had 24: how many had each at first?

15. A has two kinds of coin: it takes a pieces of the first kind to make a dollar, and b pieces of the second kind: how many pieces of each kind must be taken so that c pieces will make a dollar?

16. If 2 be added to the numerator of a certain fraction, its value will be $\frac{1}{2}$, but if 3 be added to the denominator, its value will be $\frac{1}{4}$; what is the fraction?

17. A boy bought some marbles: if he had received 5 more for the same sum, they would each have cost $\frac{1}{5}$ of a cent less; but if he had received 5 less for the same sum, they would each have cost $\frac{1}{4}$ of a cent more; how many marbles did he buy, and what did they cost?

18. A farmer bought some sheep: if he had bought 4 more for the same sum, they would each have cost $1 less; but if he had bought 5 less for the same sum, they would each have cost $2 more; how many sheep did he buy, and what did they cost?

19. Two trains, 200 feet and 152 feet long, respectively, are moving on parallel tracks, at different rates of speed though with uniform velocity; when going in opposite directions they pass each other in 3 seconds; but when moving in the same direction, the faster train passes the other in 12 seconds; what is the rate of each train per hour?

20. A farmer bought 120 acres of land for $13,200: for part of it he paid $80 an acre, and for the remainder $120 an acre; how many acres were there of each kind?

21. A and B had $2,200: A lost $\frac{1}{5}$ of his, and B lost $\frac{1}{3}$ of his, when it was found that each had the same sum; how much had each at first?

22. A certain sum of money was divided equally among a number of persons: if there had been a less persons, each would have received $\$b$ more; but if there had been c more, each would have received $\$d$ less; how many persons were there, and how much did each receive?

23. A and B are 280 rods apart: if they approach each other, they will meet in 20 minutes; but if they travel in the same direction, A will overtake B in 2 hours and 20 minutes; what is the speed of each per minute?

24. A sum of money was divided equally among a certain number of persons: if there had been 8 less persons, each would have received $3 more; but if there had been 12 more, each would have received $2 less; how many persons were there, and how much did each receive?

7. $\begin{cases} 3x + y = 6. \\ 5x + 2y = 11. \end{cases}$

8. $\begin{cases} 9x - 6y = 3. \\ 4x - y = 18. \end{cases}$

9. $\begin{cases} 8x + 3y = 5. \\ 12x + 15y = 11. \end{cases}$

10. $\begin{cases} 3x - \frac{1}{2}y = 17. \\ 2x - y = 6. \end{cases}$

11. $\begin{cases} 6x + 5y = 65. \\ 10x - 3y = 29. \end{cases}$

12. $\begin{cases} 8x - 2y = 16. \\ 5x + 7y = 43. \end{cases}$

13. $\begin{cases} 5x - 11y = -117. \\ 7x + 6y = 93. \end{cases}$

14. $\begin{cases} 5x + 4y = -17. \\ 6x - 3y = -36. \end{cases}$

15. $\begin{cases} 19x + 13y = 45. \\ 14x - 2y = 10. \end{cases}$

16. $\begin{cases} 18x - 20y = 44. \\ 17x - 15y = 26. \end{cases}$

17. $\begin{cases} \frac{2}{3}x + \frac{3}{4}y = 10. \\ \frac{1}{2}x - \frac{1}{4}y = 1. \end{cases}$

18. $\begin{cases} 10x - y = 9. \\ 10y - x = 108. \end{cases}$

MISCELLANEOUS EXAMPLES.

147. Reduce the following equations to their simplest form, and apply the method best adapted to each example:

1. Solve the equations $\begin{cases} \dfrac{\dfrac{x+y}{5} - \dfrac{x-y}{2}}{\dfrac{2x+y}{4} + \dfrac{3x-4y}{2}} = \dfrac{1}{5}. & (1) \\ x - y = 2. & (2) \end{cases}$

From (1), $$x + y - \frac{5x - 5y}{2} = \frac{2x + y}{4} + \frac{3x - 4y}{2} \qquad (3)$$

Clearing (3) of fractions,

$$4x + 4y - 10x + 10y = 2x + y + 6x - 8y \qquad (4)$$

Collecting terms, $21y - 14x = 0$ (5)

Or, $3y - 2x = 0$ (6)

Multiplying (2) by 3, $\underline{-3y + 3x = 6}$ (7)

Adding (6) and (7), $x = 6$

Substituting this value of y in (2), $y = 4$.

Ans., $x = 6$, $y = 4$.

Solve the following:

2. $$\begin{cases} \dfrac{5x-6y}{6}+\dfrac{5y}{3}=13. \\[2ex] \dfrac{5x}{6}-\dfrac{7x-4y}{3}=7. \end{cases}$$

3. $$\begin{cases} \dfrac{x}{2}+\dfrac{3y}{2}=11. \\[2ex] \dfrac{y+x}{2}+\dfrac{x-y}{2}=2. \end{cases}$$

4. $$\begin{cases} \dfrac{x+3}{2}+\dfrac{y+4}{3}=6. \\[2ex] \dfrac{2x-5}{3}-\dfrac{4y-18}{6}=0. \end{cases}$$

5. $$\begin{cases} \dfrac{x+4}{3}-\dfrac{26-y}{8}=3. \\[2ex] \dfrac{x-1}{3}+\dfrac{5(y-2)}{4}=13. \end{cases}$$

6. $$\begin{cases} \dfrac{x+2}{y+3}=\dfrac{x-2}{y-3}. \\[2ex] \dfrac{3x-5}{2y-2}=\dfrac{6x-14}{4y-2}. \end{cases}$$

7. $$\begin{cases} \dfrac{x+8}{3}+4y=28. \\[2ex] \dfrac{x+y}{2}=27-(3y+x). \end{cases}$$

8. $$\begin{cases} \dfrac{x+y}{8}+\dfrac{x-y}{6}=10. \\[2ex] \dfrac{x+y}{4}-\dfrac{x-y}{3}=10. \end{cases}$$

9. $$\begin{cases} \dfrac{x+1}{y-2}=\dfrac{x+3}{y-4}. \\[2ex] \dfrac{x-5}{y+6}=\dfrac{x-7}{y-9}. \end{cases}$$

10. $$\begin{cases} \dfrac{x+y}{x-y}=\dfrac{7}{8}. \\[2ex] \dfrac{2x-6}{3x-5}=\dfrac{4}{5}. \end{cases}$$

11. $$\begin{cases} \dfrac{x}{a}+\dfrac{y}{b}=m. \\[2ex] \dfrac{x}{b}+\dfrac{y}{a}=n. \end{cases}$$

12. $$\begin{cases} \dfrac{3x-2}{7}-\dfrac{4x-2y-1}{5}=1. \\[2ex] \dfrac{x+2}{y-2}=\dfrac{x+5}{y+6}. \end{cases}$$

13. $$\begin{cases} \dfrac{y-2}{5}-\dfrac{10-y}{3}=\dfrac{x-10}{4}. \\[2ex] \dfrac{2x+4}{3}-\dfrac{2y+x}{8}=\dfrac{y+13}{4}. \end{cases}$$

14. $\begin{cases} x(y-2)+4=y(x-3)-6. \\ 3(x-y)-y=4(x-y)-32. \end{cases}$

15. $\begin{cases} (x+2)(y-2)=19-(x-4)(y-3). \\ 3(x-2)+4(y+2)=51. \end{cases}$

16. $\left\{ \dfrac{x-2}{4}=\dfrac{y-3}{6}=\dfrac{x+y}{5}. \right.$

17. $\begin{cases} \dfrac{x+4}{2}+\dfrac{y-2}{3}=\dfrac{x+6}{2}+1. \\ \dfrac{3x-2}{5}-\dfrac{4y-6}{13}=y-x. \end{cases}$

18. $\begin{cases} \dfrac{10}{3x+5y+4}=\dfrac{12}{4x+8y+2}. \\ \dfrac{x-2}{3}=\dfrac{y-3}{4}. \end{cases}$

19. $\begin{cases} \dfrac{2x+y}{x-3y}=9. \\ \dfrac{x+4}{3}-\dfrac{x}{2}-\dfrac{y-1}{4}+\dfrac{7}{4}=\dfrac{3}{20}. \end{cases}$

20. $\begin{cases} \dfrac{\dfrac{3x+2y}{4}}{\dfrac{4x-2y}{3}}-2=7-\dfrac{\dfrac{5x-2y}{2}}{\dfrac{8x-4y}{3}}. \\ 38y=27x+218. \end{cases}$

21. $\begin{cases} \dfrac{4}{3x+4y-2}=\dfrac{3}{3y+4x-5}. \\ \dfrac{5}{6x+7y+2}=\dfrac{3}{13x-4y+7}. \end{cases}$

22. $\begin{cases} 2x+3y-6=\dfrac{2(y-x)}{3}+5\frac{1}{3}+\dfrac{x+y}{5}. \\ (x+2)(y-2)=(x-1)(y+1). \end{cases}$

23. $$\begin{cases} \dfrac{2x-y}{4}+4=\dfrac{3y-x}{7}+\dfrac{5}{2}. \\ \dfrac{4x+y}{11}+1=\dfrac{4y+x}{7}+x-5. \end{cases}$$

24. $$\begin{cases} \dfrac{x+y}{9}+\dfrac{x-y}{2}=1. \\ \dfrac{x+y}{6}-\dfrac{x-y}{5}=3\frac{2}{5}. \end{cases}$$

28. $$\begin{cases} .3x+.5y=1.45. \\ .09x+.7y=4.01. \end{cases}$$

25. $$\begin{cases} \dfrac{x+y}{x-y}=\dfrac{3}{4}. \\ \dfrac{x+y+2}{x-y-2}=\dfrac{3}{7}. \end{cases}$$

29. $$\begin{cases} .16x-.02y=.52. \\ .11x-.003y=.422. \end{cases}$$

26. $$\begin{cases} \dfrac{7x+3}{5+x-y}=\dfrac{4}{7}. \\ x+y-10=0. \end{cases}$$

30. $$\begin{cases} .5x+3y=40. \\ 8x+.33\frac{1}{3}y=68. \end{cases}$$

27. $$\begin{cases} \dfrac{3x+4}{5x-2}=\dfrac{4-3y}{6-5y}. \\ \dfrac{x}{2}+\dfrac{y+5}{3}=3\frac{1}{2}. \end{cases}$$

31. $$\begin{cases} \dfrac{.001x+.01y}{2}=2.5. \\ \dfrac{.01x+.001y}{7}=2.9. \end{cases}$$

148. Literal Simultaneous Equations.—It will generally be found most convenient in solving literal simultaneous equations to eliminate by addition or subtraction.

1. Solve the equations $\begin{cases} ax+by=c. & (1) \\ mx+ny=d. & (2) \end{cases}$

Multiplying (1) by m, $amx+bmy=cm$ (3)
Multiplying (2) by a, $amx+any=ad$ (4)
Subtracting (3) from (4), $any-bmy=ad-cm$ (5)
Whence, $y=\dfrac{ad-cm}{an-bm}$, *Ans.*
Multiplying (1) by n, $anx+bny=cn$ (6)
Multiplying (2) by b, $bmx+bny=bd$ (7)
Subtracting (7) from (6), $anx-bmx=cn-bd$ (8)
Whence, $x=\dfrac{cn-bd}{an-bm}$, *Ans.*

EXAMPLES.

Solve the following:

2. $\begin{cases} x+y=a. \\ x-y=b. \end{cases}$

3. $\begin{cases} x+y=m-n. \\ x-y=m+n. \end{cases}$

4. $\begin{cases} x+y=2m. \\ x-y=2n. \end{cases}$

5. $\begin{cases} ax+by=c. \\ bx+ay=d. \end{cases}$

6. $\begin{cases} 2x+3y=a. \\ 3x+2y=b. \end{cases}$

7. $\begin{cases} \dfrac{x}{m}-\dfrac{y}{n}=a. \\ \dfrac{x}{n}+\dfrac{y}{m}=b. \end{cases}$

8. $\begin{cases} mx+ny=0. \\ x-y=a. \end{cases}$

9. $\begin{cases} \dfrac{x}{m}+\dfrac{y}{n}=2mn. \\ x+y=mn(m+n). \end{cases}$

10. $\begin{cases} \dfrac{x}{m}-\dfrac{y}{n}=1. \\ \dfrac{x}{n}+\dfrac{y}{m}=1. \end{cases}$

11. $\begin{cases} \dfrac{x}{m+n}+\dfrac{y}{m-n}=1. \\ x-y=m. \end{cases}$

12. $\begin{cases} x+y=a+b. \\ x-y=a-b. \end{cases}$

13. $\begin{cases} \dfrac{ax}{2}+\dfrac{by}{3}=2ab. \\ \dfrac{ax}{3}+\dfrac{by}{2}=3ab. \end{cases}$

14. $\begin{cases} \dfrac{x+y}{m}+\dfrac{x-y}{n}=1. \\ x+y=m. \end{cases}$

15. $\begin{cases} (a+b)x-cy=bc. \\ x+y=a+b. \end{cases}$

16. $\begin{cases} \dfrac{ax+by}{2}=m. \\ \dfrac{ax-by}{2}=n. \end{cases}$

17. $\begin{cases} \dfrac{x}{2}+\dfrac{y}{4}=a. \\ \dfrac{x}{7}-\dfrac{y}{9}=b. \end{cases}$

18. $\begin{cases} .5x+.7y=m. \\ .05x-.9y=n. \end{cases}$

19. $\begin{cases} ax+by=\dfrac{a^2+b^2}{ab^2}. \\ bx-ay=\dfrac{a^2-b^2}{a^2b}. \end{cases}$

20. $$\begin{cases} \dfrac{x}{a+b} - \dfrac{y}{a-b} = \dfrac{1}{a-b}. \\[2ex] \dfrac{x}{a-b} + \dfrac{y}{a+b} = \dfrac{1}{a+b}. \end{cases}$$

21. $$\begin{cases} \dfrac{x+y}{a+b} + \dfrac{x-y}{a-b} = \dfrac{2}{a-b}. \\[2ex] \dfrac{x}{a} + \dfrac{y}{b} = \dfrac{a-b}{ab}. \end{cases}$$

22. $$\begin{cases} \dfrac{x+y}{m} + \dfrac{x-y}{n} = 4. \\[2ex] \dfrac{x}{m+n} + \dfrac{y}{m-n} = 2. \end{cases}$$

23. $$\begin{cases} \dfrac{x}{n} + \dfrac{y}{m} = p. \\[2ex] ax + by = c. \end{cases}$$

149. When the unknown quantities occur in the denominators, it is better, generally, not to clear of fractions.

1. Solve the equations $$\begin{cases} \dfrac{2}{x} + \dfrac{3}{y} = 4. & (1) \\[2ex] \dfrac{3}{x} + \dfrac{2}{y} = 1. & (2) \end{cases}$$

Multiplying (1) by 3, $$\frac{6}{x} + \frac{9}{y} = 12 \qquad (3)$$

Multiplying (2) by 2, $$\frac{6}{x} + \frac{4}{y} = 2 \qquad (4)$$

Subtracting (4) from (3), $$\frac{5}{y} = 10$$

Whence, $$y = \frac{1}{2}, \text{ Ans.}$$

Substituting this value of y in (1), $$\frac{2}{x} + 6 = 4$$

Whence, $$x = -1, \text{ Ans.}$$

Solve the following equations:

2. $$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=a. \\ \dfrac{1}{x}-\dfrac{1}{y}=b. \end{cases}$$

3. $$\begin{cases} \dfrac{2}{x}+\dfrac{5}{y}=14. \\ \dfrac{3}{x}-\dfrac{2}{y}=2. \end{cases}$$

4. $$\begin{cases} \dfrac{5}{x}+\dfrac{6}{y}=33. \\ \dfrac{8}{x}-\dfrac{5}{y}=9. \end{cases}$$

5. $$\begin{cases} \dfrac{7}{x}-\dfrac{3}{y}=19. \\ \dfrac{3}{x}+\dfrac{4}{y}=24. \end{cases}$$

6. $$\begin{cases} \dfrac{m}{x}+\dfrac{n}{y}=1. \\ \dfrac{n}{x}+\dfrac{m}{y}=1. \end{cases}$$

7. $$\begin{cases} \dfrac{4}{x}+\dfrac{5}{2y}=12. \\ \dfrac{3}{2x}+\dfrac{5}{3y}=18. \end{cases}$$

8. $$\begin{cases} \dfrac{9}{x}-\dfrac{5}{y}=\dfrac{2}{3}. \\ \dfrac{4}{x}+\dfrac{5}{y}=\dfrac{3}{2}. \end{cases}$$

9. $$\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}=m. \\ \dfrac{3}{x}-\dfrac{2}{y}=n. \end{cases}$$

10. $$\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=4. \\ \dfrac{b}{x}+\dfrac{a}{y}=5. \end{cases}$$

11. $$\begin{cases} \dfrac{a}{x}+\dfrac{b}{y}=m. \\ \dfrac{b}{x}+\dfrac{a}{y}=n. \end{cases}$$

12. $$\begin{cases} 3(x+3y)=2(3x-y)+38. \\ 5x-y=3(6y-5x)-36. \end{cases}$$

13. $$\begin{cases} 2x=y+3. \\ \dfrac{1}{5}(2x+5y)+1=\dfrac{2}{3}(5x-y-3). \end{cases}$$

14. $$\begin{cases} \dfrac{1}{7}(x+2)+\dfrac{1}{4}(2y-x)=3x-32. \\ \dfrac{1}{3}(2x+y+2)-\dfrac{1}{5}(2y+x-2)=3y-24. \end{cases}$$

PROBLEMS IN SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

150.—1. A bought 5 calves and 6 sheep for $89, and at the same price 7 calves and 8 sheep for $121; what was the price of each per head?

Let x = the price of a calf,
And y = the price of a sheep.
Then $5x + 6y = 89 \quad (1)$
And $7x + 8y = 121 \quad (2)$
Multiplying (1) by 7, $35x + 42y = 623 \quad (3)$
Multiplying (2) by 5, $35x + 40y = 605 \quad (4)$
Subtracting (4) from (3), $2y = 18$
$y = 9$
Substituting this value of y in (1), $5x + 54 = 89$
$5x = 35$
$x = 7$

Ans., calves, $7; sheep, $9.

2. If 6 lbs. of sugar and 10 lbs. of tea cost $6.30, and at the same price 10 lbs. of sugar and 6 lbs. of tea cost $4.10, what is the price of each per pound?

3. A hired 10 men and 8 boys for $30 a day, and at the same rate 7 men and 12 boys for $29 a day; how much did each earn in a day?

4. A drover sold 8 horses and 14 cows for $1,060, and at the same rate 12 horses and 20 cows for $1,560; what was the price of each per head?

5. A mixed sugar worth 6 cents and 9 cents a pound so as to form a mixture of 60 pounds worth 7 cents a pound; how many pounds of each kind did he take?

6. A boy bought 12 oranges and 10 bananas for 56 cents; but the price having fallen $\frac{1}{2}$, he sold 8 oranges and 3 bananas for 15 cents; required the first cost of each.

7. A bought a oranges and b apples for n cents, and at the same rate b oranges and a apples for m cents; required the price of each.

8. If the length of a rectangle be increased 5 ft. and the width 3 ft., the area will be increased 270 sq. ft.; but if the length be diminished 3 ft. and the width 5 ft., the area will be diminished 266 sq. ft. Required the dimensions.

SUGGESTION.—Let x = the length and y = the width; then $(x+5)(y+3) = xy + 270$, and $(x-3)(y-5) = xy - 266$.

9. If the length of a rectangle is increased 3 rd. and its width 2 rd., the area will be increased one acre; but if the length be diminished 2 rd. and the width 3 rd., the area will be diminished 1 acre; find the dimensions of the field.

10. If the length of a rectangular field is increased 10 rd., the area will be increased 1 acre; but if the width be increased 10 rd., the area will be increased 2 acres; find the dimensions of the field.

11. There is a number consisting of two digits: the sum of the digits is 11, and if 9 be subtracted from the number, the digits will be inverted; find the number.

12. Find a number of two digits which, being increased by the number expressed by the digits inverted, equals 77, and diminished by the same number, equals 9.

13. If 18 be subtracted from a number of two places, the digits will be inverted; but if 18 be added to the number, the sum will be 21 less than twice the number when the digits are inverted; what is the number?

14. A and B play marbles: A loses to B as many as B has, then B loses to A as many as A now has, after which it was found that each had 24; how many had each at first?

15. A has two kinds of coin: it takes a pieces of the first kind to make a dollar, and b pieces of the second kind; how many pieces of each kind must be taken so that c pieces will make a dollar?

16. If 2 be added to the numerator of a certain fraction, its value will be $\frac{1}{7}$, but if 3 be added to the denominator, its value will be $\frac{1}{8}$; what is the fraction?

17. A boy bought some marbles: if he had received 5 more for the same sum, they would each have cost $\frac{1}{5}$ of a cent less; but if he had received 5 less for the same sum, they would each have cost $\frac{1}{4}$ of a cent more; how many marbles did he buy, and what did they cost?

18. A farmer bought some sheep: if he had bought 4 more for the same sum, they would each have cost $1 less; but if he had bought 5 less for the same sum, they would each have cost $2 more; how many sheep did he buy, and what did they cost?

19. Two trains, 200 feet and 152 feet long, respectively, are moving on parallel tracks, at different rates of speed though with uniform velocity; when going in opposite directions they pass each other in 3 seconds; but when moving in the same direction, the faster train passes the other in 12 seconds; what is the rate of each train per hour?

20. A farmer bought 120 acres of land for $13,200: for part of it he paid $80 an acre, and for the remainder $120 an acre; how many acres were there of each kind?

21. A and B had $2,200: A lost $\frac{1}{5}$ of his, and B lost $\frac{1}{3}$ of his, when it was found that each had the same sum; how much had each at first?

22. A certain sum of money was divided equally among a number of persons: if there had been a less persons, each would have received \$$b$ more; but if there had been c more, each would have received \$$d$ less; how many persons were there, and how much did each receive?

23. A and B are 280 rods apart: if they approach each other, they will meet in 20 minutes; but if they travel in the same direction, A will overtake B in 2 hours and 20 minutes; what is the speed of each per minute?

24. A sum of money was divided equally among a certain number of persons: if there had been 8 less persons, each would have received $3 more; but if there had been 12 more, each would have received $2 less; how many persons were there, and how much did each receive?

25. A invested \$6,000—a part in 3% bonds, and the remainder in $4\frac{1}{2}$ per cents; how much did he put into each if he received the same income from both investments?

26. A certain sum of money, at simple interest, amounted in 3 years to \$177, and in 5 years to \$195; what was the sum, and what the rate of interest?

27. In a political convention a resolution was passed by a majority of 30 votes; if $\frac{2}{5}$ of those who voted for it had voted against it, the resolution would have been lost by 42 votes. How many voted altogether?

28. A certain sum of money at simple interest amounted in m years to \$$a$, and in n years to \$$b$; required the sum and the rate of interest.

29. A number consists of two digits: if the digits are inverted, the sum of the resulting number and the original number is 165; and if the number is divided by the sum of its digits, the quotient is $6\frac{2}{5}$; what is the number?

30. A number consists of two digits: if the number is divided by the sum of its digits, the quotient is 8 and the remainder is 4; if the digits are inverted, the quotient of the resulting number divided by the sum of its digits is 2, and the remainder is 7; what is the number?

31. In a mile race M gives N a start of 20 rods, and beats him 40 seconds; in the second heat M gives N a start of 48 seconds, and beats him 10 rods; find the rate of each per hour.

32. In a 2-mile race A gives B a start of 40 rods, and beats him 20 minutes; in the second heat A gives B a start of 21 minutes, and beats him 20 rods; required the rate of each per hour.

33. A merchant has two casks of wine. He pours from the first cask into the second as much as the second contained at first; he then pours from the second into the first as much as was left in the first; and again from the first into the second as much as was left in the second. There are now 40 gallons in each cask. How many gallons did each contain at first?

SIMPLE EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

151. When there are three or more unknown quantities and a like number of equations, the reduction may be effected as follows:

1. Solve the equations
$$\begin{cases} 2x + 3y + 4z = 20. & (1) \\ 3x + 2y + 5z = 22. & (2) \\ 5x + y + 3z = 16. & (3) \end{cases}$$

$$\begin{array}{lrcl}
\text{Multiplying (1) by 3,} & 6x + 9y + 12z & = & 60 \quad (4) \\
\text{Multiplying (2) by 2,} & \underline{6x + 4y + 10z} & = & \underline{44} \quad (5) \\
\text{Subtracting,} & 5y + 2z & = & 16 \quad (6) \\
\text{Multiplying (1) by 5,} & 10x + 15y + 20z & = & 100 \quad (7) \\
\text{Multiplying (3) by 2,} & \underline{10x + 2y + 6z} & = & \underline{32} \quad (8) \\
\text{Subtracting,} & 13y + 14z & = & 68 \quad (9) \\
\text{Multiplying (6) by 7,} & \underline{35y + 14z} & = & \underline{112} \quad (10) \\
\text{Subtracting,} & 22y & = & 44 \\
 & y & = & 2 \\
\text{Substituting in (6),} & 10 + 2z & = & 16 \\
\text{Whence,} & z & = & 3 \\
\text{Substituting in (1),} & 2x + 6 + 12 & = & 20 \\
\text{Whence,} & x & = & 1
\end{array}$$

Ans., $x = 1,\ y = 2,\ z = 3.$

EXAMPLES.

Solve the following equations:

2\.
$$\begin{cases} x + y - z = 2. \\ x - y + z = 6. \\ y - x + z = 10. \end{cases}$$

4\.
$$\begin{cases} 2x + 3z = 21. \\ 4y - z = 3. \\ 3x + 2z = 19. \end{cases}$$

3\.
$$\begin{cases} 2x + 3y + z = 17. \\ 2x + 2y + z = 14. \\ x + 3y + 2z = 19 \end{cases}$$

5\.
$$\begin{cases} 6x - y = 8. \\ 5z - 10x = 10. \\ 4y - 3x = 10. \end{cases}$$

6. $$\begin{cases} x-y=1. \\ y+z=1. \\ x-z=1. \end{cases}$$

7. $$\begin{cases} 3x+4y-z=8. \\ 4z-2y-x=7. \\ 5y-2x+7z=29. \end{cases}$$

8. $$\begin{cases} 2x+2y-z=9. \\ 3y+3z-x=33. \\ 4x+4z-y=35. \end{cases}$$

9. $$\begin{cases} 3x-5y+4z=12. \\ x-3y+7z=50. \\ x-y+z=6. \end{cases}$$

10. $$\begin{cases} x-y+z=2\tfrac{1}{2}. \\ 5x+2y+3z=23. \\ 8x+6y+5z=41. \end{cases}$$

11. $$\begin{cases} x+y+z=a. \\ x-y-z=b. \\ x+y-z=c. \end{cases}$$

12. $$\begin{cases} 3x+4y-z=20. \\ 3y+4z-x=40. \\ 3z+4x-y=30. \end{cases}$$

13. $$\begin{cases} x+y-z=0. \\ y+z-x=40. \\ z+x-y=20. \end{cases}$$

The solution can sometimes be simplified in the following way:

14. $$\begin{cases} x+y+z=9. & (1) \\ x+y+u=10. & (2) \\ x+z+u=11. & (3) \\ y+z+u=12. & (4) \end{cases}$$

Adding the equations, $3x+3y+3z+3u=42$ (5)

Whence $x+y+z+u=14$ (6)

Subtracting (4) from (6), $x=2$, *Ans.*

Subtracting (3) from (6), $y=3$, *Ans.*

Subtracting (2) from (6), $z=4$, *Ans.*

Subtracting (1) from (6), $u=5$, *Ans.*

15. $$\begin{cases} x+y=9. \\ y+z=11. \\ x+z=10. \end{cases}$$

16. $$\begin{cases} x+y+z=12. \\ x+y+u=13. \\ x+z+u=14. \\ y+z+u=15. \end{cases}$$

17. $$\begin{cases} x+y=17. \\ y+z=19. \\ z+x=18. \end{cases}$$

18. $$\begin{cases} x+y+z=a. \\ x+y+u=b. \\ x+z+u=c. \\ y+z+u=d. \end{cases}$$

19. $$\begin{cases} x+y+z+u=10. \\ x+y+z+v=11. \\ x+y+u+v=12. \\ x+z+u+v=13. \\ y+z+u+v=14. \end{cases}$$

20. $$\begin{cases} \frac{1}{x}+\frac{1}{y}=3. \quad \text{(See Art. 149.)} \\ \frac{1}{y}+\frac{1}{z}=2. \\ \frac{1}{x}+\frac{1}{z}=4. \end{cases}$$

21. $$\begin{cases} \frac{1}{x}+\frac{1}{y}=m. \\ \frac{1}{y}+\frac{1}{z}=n. \\ \frac{1}{z}+\frac{1}{u}=p. \\ \frac{1}{x}+\frac{1}{z}=q. \end{cases}$$

22. $$\begin{cases} 2x+3y+z=23. \\ 3x+y+2u=25. \\ x+3z+u=24. \\ 2y+2z+3u=36. \end{cases}$$

23. $$\begin{cases} \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3. \\ \frac{1}{x}-\frac{1}{y}+\frac{1}{z}=2. \\ \frac{1}{x}+\frac{1}{y}-\frac{1}{z}=1. \end{cases}$$

24. $$\begin{cases} \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \\ \frac{x}{a}+\frac{y}{c}+\frac{z}{b}=1. \\ \frac{x}{b}+\frac{y}{a}+\frac{z}{c}=1. \end{cases}$$

25. $$\frac{x+y}{5}=\frac{x+z}{6}=\frac{z+y}{7}=2.$$

26. $$\frac{2x-z}{3}=\frac{5y-3z}{9}=\frac{10x-7y}{8}=2.$$

27. $$\frac{x+y}{a}=\frac{x+z}{b}=\frac{z+y}{c}=2.$$

28. $$\frac{x-y}{m}=\frac{z-x}{n}=\frac{2z+y}{p}=3.$$

29. $$xyz=2(xz+xy-yz)=3(xz+yz-xy)=4(xy+yz-xz).$$

30. $$\frac{x-y}{a}=\frac{y-z}{b}=\frac{x+z}{c}=a.$$

PROBLEMS PRODUCING EQUATIONS CONTAINING ONE OR MORE THAN ONE UNKNOWN QUANTITY.

152.—1. M hired 8 men, 10 women, and 12 boys for $222 a week; and, at another time, 19 men, 8 women, and 11 boys for $333 a week. What was the daily wages of each if he paid a man, a woman, and a boy $4 a day?

2. Find three numbers, such that the first plus $\frac{1}{2}$ of the second, the second plus $\frac{1}{3}$ of the third, and the third minus $\frac{2}{3}$ of the first may each be equal to 60.

3. Find four numbers, such that the first with $\frac{1}{2}$ of the other three, the second with $\frac{1}{3}$ of the other three, the third with $\frac{1}{4}$ of the other three, and the fourth with $\frac{1}{5}$ of the other three may each be equal to 37.

4. Divide 84 into four such parts that if the first be divided by 2, the second by 3, the third by 4, and the fourth by 5, the quotients will all be equal.

5. A and B earn $90 in 10 days, B and C earn $49 in 7 days, and A and C earn $72 in 12 days; required the daily wages of each.

6. A invests $7,620 in bonds: he buys 4% bonds at 90, 5% bonds at 105, and 6% bonds at 120; how much must he expend in each kind to obtain the same income from each investment?

7. The sum of the digits of a number of three figures is 6: the digit in the ten's place is one half of the sum of the other two, and the number expressed by the two left-hand digits is four times the unit's digit; what is the number?

8. If a number of three places is divided by the sum of its digits, the quotient is $22\frac{3}{5}$; the digit in the unit's place is three times that in the hundredth's place; and if 396 be added to the number, the digits will be reversed. Required the number.

9. A and B can do a piece of work in 12 days, B and C in 16 days, and A and C in 20 days; how long will it take each to do it?

10. A, B, and C can do a piece of work in 20 days; B, C, and D in 24 days; A, C, and D in 30 days; and A, B, and D in 36 days. How long will it take each to do it?

11. A certain sum of money consists of dollars, dimes, and half-dimes: it is worth as many dimes as there are pieces of money, and as many dollars as there are dimes plus one; how many pieces are there of each kind if the number of half-dimes is nine times the number of dimes?

12. A certain sum of money consists of quarter-dollars, dimes, half-dimes, and three-cent pieces: it is worth as many dimes as there are pieces of money; it is worth as many quarters as there are half-dimes; the number of half-dimes plus the number of three-cent pieces is 12 more than the number of quarters and dimes, and the sum of the quarters and three-cent pieces is 48; what is the sum of the money?

13. A gave B and C as much money as they already had; B gave A and C as much as each of them then had; and C gave A and B as much as each of them then had,—after which it was found that each of them had $60. How much had each at first?

14. M, N, and O agreed to plow a field: the first day M worked 3 hours, N 5 hours, and O $3\frac{1}{3}$ hours, and together they plowed 2 acres; the second day M worked 4 hours, N 6 hours, and O 8 hours, and together they plowed 3 acres; the third day M worked 5 hours, N 9 hours, and O 10 hours, and together they plowed 4 acres. In what time could each alone plow an acre?

15. A dairyman has two cans of milk: he pours from the first into the second as much as the second contained at first; he then pours from the second into the first as much as was left in the first; and again from the first into the second as much as was left in the second. How many gallons did each can contain at first if there are now 24 gallons in each?

Therefore, to raise a monomial to any power,

RULE.

Raise the coefficient to the required power, multiply the exponent of each letter by the exponent of the power, and prefix the proper sign.

EXAMPLES.

By inspection, find the values of the following:

3. $(2a^3b^2)^2$.
4. $(-3ab^4)^2$.
5. $(4a^3x^2)^3$.
6. $(-3a^2x^4)^3$.
7. $(2x^2y^3z)^3$.
8. $(-3x^ny^{2m})^4$.
9. $(-2x^nb^mc^2)^5$.
10. $(x^{2n}y^{3m}c^{n^2})^5$.
11. $(a^3b^2c^4)^6$.
12. $(-a^tb^s)^3$.
13. $(-a^nb^m)^n$.
14. $(-2a^{n^2}b^m)^n$.
15. $(-4a^mx^n)^{mn}$.
16. $(a^{n^2}b^{2n})^n$.
17. $(-a^2b^3)^{2n+1}$.
18. $(-n^2m^2)^n$.
19. $(2a^3b)^n$.
20. $(-a^{2n}m)^2$.
21. $(-6x^ry^t)^3$.
22. $(x^ny^m)^4$.
23. $(-3a^2y^3)^5$.
24. $(-a^nb^n)^{2n}$.
25. $(-x^mb^n)^{3n}$.
26. $(-2ab^m)^n$.

From the multiplication of fractions it is readily seen that a fraction is raised to any power by *raising both numerator and denominator to the required power.*

For example, $\left(-\dfrac{3x^m}{2y^n}\right)^3 = -\dfrac{27x^{3m}}{8y^{3n}}$.

By inspection, find the values of the following:

27. $\left(\dfrac{ab}{c}\right)^2$.
28. $\left(-\dfrac{m^2n}{x}\right)^3$.
29. $\left(-\dfrac{-x^2}{y}\right)^3$.
30. $\left(-\dfrac{-a^2}{-c}\right)^3$.
31. $\left(-\dfrac{m^2x^2}{y^3}\right)^n$.
32. $\left(\dfrac{4x^m}{y^n}\right)^3$.
33. $\left(\dfrac{x^x}{y^y}\right)^{xy}$.
34. $\left(\dfrac{x^2}{y^3}\right)^n$.
35. $\left(\dfrac{x^x}{2}\right)^2$.
36. $\left(\dfrac{y^{n^2}}{4}\right)^n$.
37. $\left(-\dfrac{x^2}{y^3}\right)^m$.
38. $\left(-\dfrac{3a^2y}{4m^2x}\right)^4$.

SPECIAL METHODS OF SQUARING POLYNOMIALS.

FIRST METHOD.

157. Polynomials may be squared by multiplication; but the result may be written without performing the actual multiplication.

1. Square $a + b + c$.

By multiplication,

$$\begin{array}{llllll}
a\,+ & b\,+ & c & & & \\
a\,+ & b\,+ & c & & & \\
\hline
a^2+ & ab+ & ac & & & \\
 & ab & & +\,b^2+ & bc & \\
 & & +\,ac & & +\,bc+ & c^2 \\
\hline
a^2+ & 2ab+ & 2ac+ & b^2+ & 2bc+ & c^2
\end{array}$$

This result may be arranged thus:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

From this arrangement we derive the following

RULE.

The square of a polynomial equals the square of each term, together with twice the product of the terms taken two and two.

2. Square $a - b + c - d$.

$$(a - b + c - d)^2 = a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2ad - 2bc + 2bd - 2cd.$$

EXAMPLES.

Square:

3. $a + b + c + d$.
4. $a - b - c + d$.
5. $m + n + o - l$.
6. $a + b - c - d$.
7. $a + b - c - 1$.
8. $1 + x + x^2 + x^3$.
9. $2a + 3b + 4c + d$.
10. $2x + 3y - 2z + v$.
11. $x^2 - 2xy + y^2 + 3z$.
12. $l - m + p - r - s$.
13. $a + l + g + e + b + r + a$.
14. $r + e + p + u + b + l + i + c$.

15. $2a + 3b - 2c - 1.$

16. $x^2 + x^3 + x^4 + x^5.$

17. $m^2 - n^2 - n - m.$

18. $1 - x + 2x^2 - 3x^3.$

19. $x^2 - x - 5 - \frac{2}{x}.$

20. $a + b + c \ldots\ldots l.$

SECOND METHOD.

158. Arranging the terms of the result in Ex. 1, Art. 157, in another order, and factoring, we have

$$(a + b + c)^2 = a^2 + 2ab + b^2 + 2(a + b)c + c^2.$$

In this method square the first two terms, thus,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Then regard the first two terms as one, and the third term as the second term, and we have

$$\begin{aligned}[(a + b) + c]^2 = a^2 &+ 2ab + b^2 \\ &+ 2(a + b)c + c^2.\end{aligned}$$

1. Square $(a + b - c - d)$.

$$(a + b - c - d)^2 = a^2 + 2ab + b^2 - 2(a + b)c + c^2 - 2(a + b - c)d + d^2.$$

From these examples we derive the following

RULE.

The square of a polynomial equals the square of the first term, plus (or minus) twice the product of the first term into the second, plus the square of the second, plus (or minus) twice the sum of the first two terms into the third, plus the square of the third, etc.

Square by the second method—

2. $t + u.$

3. $h + t + u.$

4. $T + h + t + u.$

5. $a + b + c + d + e.$

6. $a - b + c - d.$

7. $m + n - p + q.$

8. $a - b + c - d - e.$

9. $x + 2x^2 + 3x^3 + 4x^4.$

10. $a + b - c + e - f - g.$

11. $2a - 3b - c + 3d - 4e.$

12. $l - m - n + p - q - r.$

13. $A + Bx + Cx^2 + Dx^3.$

THIRD METHOD.

159. A third form is derived from the second method by factoring.

$$\text{Thus, } (a + b + c)^2 = a^2 + (2a + b)b + [2(a + b) + c]c.$$

1. Square $a + b - c - d$.

$$(a + b - c - d)^2 = a^2 + (2a + b)b - [2(a + b) - c]c - [2(a + b - c) - d]d.$$

Let the pupil derive the rule.

NOTE.—Each of these methods has its peculiar advantage: the first method is the most expeditious, the second method is the most convenient form for general application, and the third method is the form used in extracting square root.

Square by the third method—

2. $a + b + c$.
3. $a + b - c$.
4. $a - 2b - 3c$.
5. $2m - 3n + 4p$.
6. $x - 2x^2 - 3x^3$.
7. $T + h + t + u$.
8. $m - n - o - p$.
9. $2x - 3y - 4z + 5v$.
10. $1 + x + x^2 + x^3 + x^4$.
11. $A + Bx + Cx^2 + Dx^3$.

SPECIAL METHODS OF CUBING POLYNOMIALS.

FIRST METHOD.

160.—1. Cube $a + b$.

By squaring (ART. 157),

$$\begin{array}{rl} (a + b)^2 = & a^2 + 2ab + b^2 \\ & \underline{a + b} \\ & a^3 + 2a^2b + ab^2 \\ & \underline{\ a^2b + 2ab^2 + b^3} \\ & a^3 + 3a^2b + 3ab^2 + b^3, \textit{ Ans.} \end{array}$$

By actual multiplication,

$$(a + b + c)^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3(a + b)^2c + 3(a + b)c^2 + c^3.$$

From this result we derive the following

RULE.

The cube of a polynomial having positive terms equals the cube of the first term, plus three times the square of the first into the second, plus three times the first into the square of the second, plus the cube of the second, plus three times the square of the sum of the first and second into the third, plus three times the sum of the first and second into the square of the third, plus the cube of the third; and so on.

2. Cube $2a - 3b$.

$$(2a - 3b)^3 = (2a)^3 - 3(2a)^2(3b) + 3(2a)(3b)^2 - (3b)^3 = 8a^3 - 36a^2b + 54ab^2 - 27b^3, \textit{ Ans.}$$

3. Cube $a + 2b - 3c$.

$$(a + 2b - 3c)^3 = a^3 + 3a^2(2b) + 3a(2b)^2 + (2b)^3 - 3(a + 2b)^2(3c) + 3(a + 2b)(3c)^2 - (3c)^3$$

$$= a^3 + 6a^2b + 12ab^2 + 8b^3 - 9(a + 2b)^2c + 27(a + 2b)c^2 - 27c^3.$$

EXAMPLES.

Cube by the first method—

4. $x + y$.
5. $m + n$.
6. $a - b$.
7. $m - n$.
8. $a + 1$.
9. $1 - x$.
10. $2a + 3b$.
11. $3m - 2n$.
12. $x - 3y$.
13. $2a - 4b$.
14. $x + 3$.
15. $ab + c$.
16. $a^2 + b$.
17. $3m + n^2$.
18. $2x^2 - 3y^2$.
19. $a + b + c$.
20. $m + n + p$.
21. $x + y + z$.
22. $a + b - c$.
23. $3x^2 + 2y^2$.
24. $a + b + c + d$.
25. $a - b + c - d$.
26. $a + 2b + 3c + d$.
27. $A + Bx + Cx^2$.
28. $1 + x - y + z$.

SECOND METHOD.

161. A second form is derived from the first method by factoring.

$$
\begin{aligned}
\text{Thus, } (a+b)^3 &= a^3 + (3a^2 + 3ab + b^2)b.\\
(a+b+c)^3 &= a^3 + (3a^2 + 3ab + b^2)b\\
&\quad + [3(a+b)^2 + 3(a+b)c + c^2]c.\\
(a+b+c+d)^3 &= a^3 + (3a^2 + 3ab + b^2)b\\
&\quad + [3(a+b)^2 + 3(a+b)c + c^2]c\\
&\quad + [3(a+b+c)^2 + 3(a+b+c)d + d^2]d.
\end{aligned}
$$

Let the pupil derive the rule.

NOTE.—The first method is the most convenient form for general application, and the second is the form used in extracting cube root.

Cube by the second method—

1. $t + u$.
2. $h + t + u$.
3. $T + h + t + u$.
4. $a + b - c$.
5. $a - b - c$.
6. $m + n + p - d$.
7. $a + 2b + 3c$.
8. $a - b + c - d$.

THE BINOMIAL THEOREM.

162. The **Binomial Theorem** is a general method for raising a binomial to any required power. It is a ready method of obtaining the powers of a binomial without the tedious process of multiplication.

By actual multiplication we obtain

$$
\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2\\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3\\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4, \text{ etc., etc.}
\end{aligned}
$$

$$
\begin{aligned}
(a-b)^2 &= a^2 - 2ab + b^2\\
(a-b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3\\
(a-b)^4 &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4, \text{ etc., etc.}
\end{aligned}
$$

Examining these results, we derive the following laws:

1. *The* **Number of Terms** *is one more than the exponent of the power.*

2. *The* **Signs** *of the terms are alternately positive and negative when the second term of the binomial is negative.*

3. *The* **Exponent** *of the leading letter in the first term is the same as the exponent of the power, and decreases by* 1 *in each succeeding term. The exponent of the second letter is* 1 *in the second term, and increases by* 1 *in each succeeding term.*

4. *The* **Coefficient** *of the first term is* 1, *and of the second term is the same as the exponent of the power. The coefficient of any term, multiplied by the exponent of the leading letter in that term and divided by the number of the term, equals the coefficient of the next term.*

NOTE 1.—The sum of the exponents of a and b in any term is the same as the exponent of the power.

NOTE 2.—The coefficients and exponents of the last half of the terms when *even*, or of the terms after the middle when *odd*, are the same as the first half, inversely. We may therefore write the last half of the development without actual calculation.

EXAMPLES.

1. Expand $(a+b)^6$.

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

Expand—

2. $(a+y)^4$.
3. $(a+x)^5$.
4. $(m+n)^6$.
5. $(m-n)^7$.
6. $(x+y)^8$.
7. $(x-y)^6$.
8. $(m-n)^8$.
9. $(a-b)^9$.
10. $(a+x)^{10}$.
11. $(m-n)^{10}$.
12. $(c-1)^8$.
13. $(b-1)^7$.

14. Expand $(1-x)^5$.

$$(1-x)^5 = 1^5 - 5\cdot 1^4x + 10\cdot 1^3x^2 - 10\cdot 1^2x^3 + 5\cdot 1x^4 - x^5.$$
$$= 1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5.$$

15. $(1-a)^6$. 17. $(1+x)^8$. 19. $(1-x)^8$.

16. $(1-n)^7$. 18. $(1+a)^7$. 20. $(1+m)^6$.

21. Expand $(a+b)^n$, giving the first four and last four terms.

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 +, \text{etc.}$$

The expression $\underline{|3}$ is read *factorial three*, and means $1 \times 2 \times 3$; also,

$$\underline{|4} = 1 \times 2 \times 3 \times 4; \quad \underline{|n} = 1 \times 2 \times 3 \times 4 \times \ldots\ldots n.$$

Hence, the result is written

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{\underline{|2}}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}b^3 +, \text{etc.}$$

To find the last four terms, change the order of the letters and expand.

$$(b+a)^n = b^n + nb^{n-1}a + \frac{n(n-1)}{\underline{|2}}b^{n-2}a^2 + \frac{n(n-1)(n-2)}{\underline{|3}}b^{n-3}a^3 +, \text{etc.}$$

Expand the following, giving first five and last five terms:

22. $(a+x)^m$. 24. $(a-b)^n$. 26. $(r-s)^t$.

23. $(m+n)^x$. 25. $(a-x)^m$. 27. $(a-b)^x$.

The following method of expanding binomials whose terms have coefficients will be found of great value:

28. Expand $(3a+2b)^4$.

Indicating the operation, we have $(3a+2b)^4 = (3a)^4 + \frac{4}{1}(3a)^3(2b)$

$$+ \frac{4 \cdot 3}{1 \cdot 2}(3a)^2(2b)^2 + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}(3a)(2b)^3 + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}(2b)^4.$$

By examining this result, we see that the *second* term can be obtained by multiplying the *first* term by $\frac{2b}{3a}$ and $\frac{4}{1}$; the *third*, by multiplying the *second* term by $\frac{2b}{3a}$ and $\frac{3}{2}$; the *fourth* term, by multiplying the third term by $\frac{2b}{3a}$ and $\frac{2}{3}$, etc.

Hence, any term can be obtained by multiplying the preceding term by $\frac{2b}{3a}$ (that is, the second term of the binomial divided by the first), and in succession by $\frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}$.

$$\begin{aligned}
\text{Thus, 1st term} &= (3a)^4 = 81a^4.\\
\text{2d term} &= 81a^4 \times \frac{2b}{3a} \times \frac{4}{1} = 216a^3b.\\
\text{3d term} &= 216a^3b \times \frac{2b}{3a} \times \frac{3}{2} = 216a^2b^2.\\
\text{4th term} &= 216a^2b^2 \times \frac{2b}{3a} \times \frac{2}{3} = 96ab^3.\\
\text{5th term} &= 96ab^3 \times \frac{2b}{3a} \times \frac{1}{4} = 16b^4.
\end{aligned}$$

Hence, $(3a + 2b)^4 = 81a^4 + 216a^3b + 216a^2b^2 + 96ab^3 + 16b^4$.

29. Expand $(2a^2 - 3b^3)^5$.

$$\begin{aligned}
\text{1st term} &= (2a^2)^5 = 32a^{10}.\\
\text{2d term} &= 32a^{10} \times \frac{3b^3}{2a^2} \times \frac{5}{1} = 240a^8b^3.\\
\text{3d term} &= 240a^8b^3 \times \frac{3b^3}{2a^2} \times \frac{4}{2} = 720a^6b^6.\\
\text{4th term} &= 720a^6b^6 \times \frac{3b^3}{2a^2} \times \frac{3}{3} = 1080a^4b^9.\\
\text{5th term} &= 1080a^4b^9 \times \frac{3b^3}{2a^2} \times \frac{2}{4} = 810a^2b^{12}.\\
\text{6th term} &= 810a^2b^{12} \times \frac{3b^3}{2a^2} \times \frac{1}{5} = 243b^{15}.
\end{aligned}$$

Hence,

$(2a^2 - 3b^3)^5 = 32a^{10} - 240a^8b^3 + 720a^6b^6 - 1080{}^4b^9 + 810a^2b^{12} - 243b^{15}$.

NOTE.—The signs are the same as stated in the *second law*, ART. 162.

Expand—

30. $(2a + 4b)^4$.
31. $(3x + 5y)^4$.
32. $(3a - 2c)^4$.
33. $(3a^2 + 4b^3)^3$.
34. $(4a - 3a^2x^3)^6$.
35. $\left(x - \frac{2}{x}\right)^5$.
36. $(3x^2 - 5y^2)^4$.
37. $(4a^2x + 5ax^2)^5$.
38. $(3m^2 - 4n^3)^5$.
39. $(\frac{1}{2}a - \frac{2}{3}x)^6$.
40. $(x^2 - 2a)^8$.
41. $\left(2x - \frac{1}{x}\right)^6$.
42. $(2a^2x - 3b^2)^4$.
43. $(2m^3 - 3n^2)^5$.
44. $(3 + 6b^2)^7$.
45. $(2a^2 - 3x^2)^4$.
46. $(3x^2 + 4y)^6$.
47. $\left(\frac{x}{a} - \frac{2a}{3x}\right)^5$.

EVOLUTION.

163. **Evolution** is the process of finding any required root of a quantity.

164. Review the definition—

1. Of **Radical Sign** (ART. 8).
2. Of **Root of a Quantity** (ART. 8).
3. Of **Index of the Root** (ART. 8).

165. A **Fractional Exponent** is also used to indicate a root of a quantity. The numerator indicates a power, and the denominator a root, of the quantity.

Thus, $a^{\frac{1}{2}} = \sqrt{a}$ indicates the square root of a.

$a^{\frac{2}{3}} = \sqrt[3]{a^2}$ indicates the cube root of a^2.

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$ indicates the nth root of the mth power.

166. A **Perfect Power** is a quantity whose required root can be exactly obtained. An **Imperfect Power** is a quantity whose required root can only be obtained approximately.

PRINCIPLE I.

The odd roots of a quantity have the same sign as the quantity itself.

Since $(+a)^3 = +a^3$, then $\sqrt[3]{+a^3} = +a$;

and $(-a)^5 = -a^5$, then $\sqrt[5]{-a^5} = -a$.

PRINCIPLE II.

The even roots of a positive quantity are either positive or negative.

$$\left.\begin{array}{l}\text{Since } (+a)^2 \\ \text{or } \quad (-a)^2\end{array}\right\} = +a^2, \text{ then } \sqrt{+a^2} = \left\{\begin{array}{l}+a \text{ or} \\ -a.\end{array}\right.$$

PRINCIPLE III.

The even roots of a negative quantity are impossible.

This is evident, since no quantity raised to an even power can produce a negative result.

EVOLUTION OF MONOMIALS.

167.—1. Find the cube root of $27a^6b^3c^9$.

Since in cubing a monomial we cube the coefficient and multiply the exponents of the letters by 3, then to extract the cube root of a quantity we must take the cube root of the coefficient and divide the exponent of the letters by 3.

Hence $\sqrt[3]{27a^6b^3c^9} = 3a^2bc^3$.

Therefore, to find any root of a monomial,

RULE.

Extract the required root of the coefficient, divide the exponent of each letter by the index of the root, and prefix to the result the proper sign.

NOTE 1.—If the coefficient of the given quantity is not an exact power of the given root, indicate the result; thus,

$$\sqrt{3a^2} = 3^{\frac{1}{2}}a,\ \sqrt[3]{4a^3} = 4^{\frac{1}{3}}a,\ \sqrt[n]{4a^{2n}} = 4^{\frac{1}{n}}a^2, \text{ etc.}$$

NOTE 2.—To find any root of a fraction, extract the required root of both terms.

EXAMPLES.

Find the values of the following:

2. $\sqrt{25x^6y^4z^2}$.
3. $\sqrt[3]{-27a^3b^6c^3}$.
4. $\sqrt{49m^4n^8p^{10}}$.
5. $\sqrt[3]{19x^6y^9z^{12}}$.
6. $\sqrt[3]{-64m^3n^6x^{15}}$.
7. $\sqrt[4]{m^8n^{12}p^{16}}$.
8. $\sqrt[4]{10a^4b^8c^{12}}$.
9. $\sqrt[4]{1296x^4y^{16}z^{20}}$.
10. $\sqrt[5]{-64m^5n^{10}a^{15}}$.
11. $\sqrt[3]{-512a^{12}n^9m^6y^3}$.
12. $\sqrt{a^{2n}b^{4n}c^{5n}}$.
13. $\sqrt[n]{4a^{6n}b^nc^{3n}}$.
14. $\sqrt{\frac{4}{9}b^2c^2m^n}$.
15. $\sqrt[n]{a^{2n}b^{4m-6n}c^{3n}}$.
16. $\sqrt[n]{x^{2n-1}y^{3n-2}z^{4n-3}}$.
17. $\sqrt[6]{a^{6m}b^{6n}c^{12n}d^n}$.

18. $\sqrt{\dfrac{16x^2y^4}{25n^2}}$.　20. $\sqrt[3]{-\dfrac{a^{12}b^{15}}{125x^6}}$.　22. $\sqrt[5]{\dfrac{-a^5b^{10}}{32x^5y^n}}$.

19. $\sqrt[3]{\dfrac{8m^3n^6}{27a^9b^{12}}}$.　21. $\sqrt[4]{\dfrac{16m^4n^8}{81x^8y^{12}}}$.　23. $\sqrt[4]{\dfrac{a^8b^{12}}{625m^4}}$.

24. Show that $\sqrt{\sqrt[3]{64}} = \sqrt[3]{\sqrt{64}}$.

25. Show that $\sqrt[3]{\sqrt[4]{a^{24}b^{48}}} = \sqrt[4]{\sqrt[3]{a^{24}b^{48}}}$.

26. Show that $\sqrt[4]{\sqrt[5]{a^{40}b^{60}c^{80}}} = \sqrt[5]{\sqrt[4]{a^{40}b^{60}c^{80}}}$.

27. Show that $\sqrt[m]{\sqrt[n]{a^{2mn}b^{3mn}}} = \sqrt[n]{\sqrt[m]{a^{2mn}b^{3mn}}}$.

From Examples 24–27 it is seen that in general the *m*th root of the *n*th root equals the *n*th root of the *m*th root.

SQUARE ROOT OF POLYNOMIALS.

168.—1. Find the square root of $a^2 + 2ab + b^2$.

$$\begin{array}{r|l|l}
 & a^2 + 2ab + b^2 & a + b, \\
 & a^2 & \textit{Ans.} \\ \hline
2a + b & 2ab + b^2 & \\
 & 2ab + b^2 & \\ \hline
\end{array}$$

From the method of squaring a polynomial (ART. 157), the first term, a^2, is the square of the first term of the root; hence the first term of the root is the square root of a^2, which is a. Subtracting its square from the polynomial, we have $2ab + b^2$ remaining.

The first term of this remainder is equal to twice the first term of the root into the second; hence we can find the second term of the root by dividing $2ab$ by $2a$, which gives b. Adding b to $2a$ gives $2a + b$, the complete divisor. Multiplying $2a + b$ by b and subtracting, there is no remainder. Hence, the square root of $a^2 + 2ab + b^2$ is $a + b$.

Therefore, to extract the square root of a polynomial we have the following

RULE.

Arrange the terms according to the powers of some letter.

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the polynomial.

Divide the first term of the remainder by twice the root already found, and annex the quotient to the root, and also to the trial divisor to form the complete divisor. Multiply the complete divisor by the second term of the root, and subtract the product from the remainder.

Continue in this manner until all the terms of the root are found.

2. Find the square root of $4x^4 - 12x^3 + 13x^2 - 6x + 1$.

$$\begin{array}{l|l|l}
 & 4x^4 - 12x^3 + 13x^2 - 6x + 1 & 2x^2 - 3x + 1,\ \textit{Ans.} \\
 & 4x^4 & \\ \hline
4x^2 - 3x & -12x^3 + 13x^2 & \\
 & -12x^3 + \ \ 9x^2 & \\ \hline
4x^2 - 6x + 1 & \qquad 4x^2 - 6x + 1 & \\
 & \qquad 4x^2 - 6x + 1 & \\ \hline
\end{array}$$

The square root of $4x^4$, which is $2x^2$, is the first term of the root. Subtracting $4x^4$ from the polynomial, we have $-12x^3$ for the first term of the remainder. Dividing this by twice the first term of the root, $4x^2$, we obtain $-3x$, the second term of the root, which, added to $4x^2$, gives the complete divisor, $4x^2 - 3x$. Multiplying this divisor by $-3x$, and subtracting the product from the remainder, we have $4x^2$ for the first term of the remainder. Dividing this by twice the first term of the root, $4x^2$, we obtain 1, the third term of the root, which, added to twice the root already found, gives the complete divisor, $4x^2 - 6x + 1$. Multiplying this by 1, and subtracting the product from the remainder, there is no further remainder. Hence, $2x^2 - 3x + 1$ is the required square root.

EXAMPLES.

Find the square root of the following:

3. $x^2 + 2x + 1$.
4. $x^2 + 2xy + y^2$.
5. $a^2 + 4a + 4$.
6. $9a^2 - 12ac + 4c^2$.
7. $\frac{m^2}{9} - mn + \frac{9n^2}{4}$.
8. $a^2 + ax + \frac{1}{4}x^2$.
9. $a^{2n} - 2a^nb^n + b^{2n}$.
10. $x^{4n} + 6x^{2n}y^{2n} + 9y^{4n}$.
11. $x^4 + x^3 + \frac{3}{4}x^2 + \frac{1}{4}x + \frac{1}{16}$.
12. $m^2 + 2m - 1 - \frac{2}{m} + \frac{1}{m^2}$.
13. $x^4 + x^2 + 4 - 2x^3 - 4x^2 + 4x$.
14. $a^2 + x^2 + z^2 - 2ax - 2az + 2xz$.
15. $a^2 - 4ac + 4c^2 + 6a - 12c + 9$.
16. $m^4 + 4m^3n + 10m^2n^2 + 12mn^3 + 9n^4$.
17. $9a^2 + 16b^2 + 25c^2 + 24ab + 30ac + 40bc$.
18. $10n^2 + 25n^4 - 20n^3 + 1 - 4n + 16n^6 - 24n^5$.

19. $m^6 - 2m^5 - m^4 + 3m^2 + 2m + 1$.

20. $1 - 4n + 10n^2 - 20n^3 + 25n^4 - 24n^5 + 16n^6$.

21. $a^2 - 4ab + 4b^2 + 6ac - 12bc + 9c^2 - 8ad + 16bd - 24cd + 16d^2$.

22. $a^2 - 6ab + 9b^2 + 8ax - 24bx + 16x^2 - 4ay + 12by - 16xy + 4y^2$.

23. $1 + x$.

$$
\begin{array}{r|l}
 & 1 + x \;\big|\; 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}, \text{ etc., } \textit{Ans.} \\
 & 1 \\
\hline
2 + \frac{x}{2} & x \\
 & x + \frac{x^2}{4} \\
\hline
2 + x - \frac{x^2}{8} & -\frac{x^2}{4} \\
 & -\frac{x^2}{4} - \frac{x^3}{8} + \frac{x^4}{64} \\
\hline
2 + x - \frac{x^2}{4} + \frac{x^3}{16} & \frac{x^3}{8} - \frac{x^4}{64}, \textit{ Remainder.}
\end{array}
$$

Find four terms of the square root of—

24. $1 - x$.

25. $a + b$.

26. $a^2 - 2b$.

27. $1 + x^2$.

28. $a^2 - b^2$.

29. $4x^2 + 2y$.

SQUARE ROOT OF NUMBERS.

169. Algebra furnishes a very satisfactory explanation of the **Square Root of Numbers**.

PRINCIPLE I.

The square of a number consists of twice as many figures as the number, or twice as many less one.

$1^2 = 1$, $10^2 = 100$, $100^2 = 10000$,

$9^2 = 81$, $99^2 = 9801$, $999^2 = 998001$.

If we square 1, the smallest number of one figure, and 9, the largest number of one figure, the square consists of *twice as many figures as the number, or twice as many less one.* In the same way we find the principle true for numbers of two and three figures; hence, we infer that it is generally true.

PRINCIPLE II.

If we point off a number into periods of two figures each, beginning at the right, the number of full periods, together with the partial period at the left if there be one, will equal the number of figures in the square root. This is evident from Prin. 1.

PRINCIPLE III.

The square of a decimal contains twice as many figures as the decimal itself.

$$.1^2 = .01, \qquad .9^2 = .81, \qquad .01^2 = .0001, \qquad .99^2 = .9801.$$

It is thus seen that the square of a decimal of one place occupies two places, and the square of a decimal of two places occupies four places; hence, we infer that the principle is generally true.

PRINCIPLE IV.

*If we represent the units by **u**, tens by **t**, hundreds by **h**, and thousands by **T**, we will obtain the following formulas:*

$$(t + u)^2 = t^2 + 2tu + u^2.$$

$$(h + t + u)^2 = h^2 + 2ht + t^2 + 2(h + t)u + u^2.$$

$$(T + h + t + u)^2 = T^2 + 2Th + h^2 + 2(T + h)t + t^2 + 2(T + h + t)u + u^2.$$

1. Find the square root of 9025.

$$\begin{array}{lr|l}
t^2 + 2tu + u^2 = & 90'25 & 90 \\
t^2 = 90^2 = & \underline{81\ 00} & \underline{\ \ 5} \\
\hline
(2t + u)u = & 9\ 25 & 95,\ \textit{Ans.} \\
2t = 2 \times 90 = 180 & & \\
\underline{u = \qquad\qquad 5} & & \\
(2t + u)u = 185 \times 5 = 925 & &
\end{array}$$

Separating the number into periods of two figures each, we see that the root will contain two figures (Prin. 2); hence, the root consists of *tens* and *units*, and the number consists of $t^2 + 2tu + u^2$.

The greatest number of *tens* whose square is contained in 9025 is 9 *tens*, or 90. Subtracting the square of 90 from 9025, we have 925 remaining, which equals $2tu + u^2$. Now, since $2tu$ is generally very much greater than u^2, 925 consists principally of $2tu$; hence, we can find the units by dividing 925 by $2t$, or 180; dividing 925 by 180 gives 5, the unit's figure, and $(2t + u)u = (180 + 5)5$, or 925; subtracting, there is no remainder. Hence the square root of 9025 is 95.

In practice we *omit the ciphers* in the root, and write it simply 95.

Therefore, to extract the square root of a number, we have the following

RULE.

Separate the number into periods of two figures each, beginning at unit's place.

Find the largest number whose square is contained in the left-hand period, write it as the first figure of the root, subtract its square from the number, and bring down the next period for a new dividend.

Divide the new dividend, omitting the last figure, by twice the root already found, and annex the quotient to the root and also to the divisor.

Multiply the complete divisor by the second term of the root and subtract the product from the dividend.

Thus continue until all the periods are used.

EXAMPLES.

2. Find the square root of 119025.

$$\begin{array}{lll|r|l}
h^2 + 2ht + t^2 + 2(h+t)u + u^2 & & = & 11'90'25 & 300 \\
h^2 = & 300^2 & = & 9\ 00\ 00 & 40 \\
 & & & & \underline{\ \ 5} \\
\hline
(2h+t)\,t + 2(h+t)u + u^2 & & = & 2\ 90\ 25 & 345,\ \textit{Ans.} \\
\quad 2h = & 2 \times 300 & = 600 & & \\
\quad\quad + t = & & \ \ 40 & & \\
\hline
(2h+t)t = & & 640 \times 40 & 2\ 56\ 00 & \\
\hline
[2(h+t)+u]u = & & & 34\ 25 & \\
\quad 2(h+t) = & 2 \times 340 & = 680 & & \\
\quad\quad + u = & & \ \ 5 & & \\
\hline
[2(h+t)+u]u = & & 685 \times 5 = & 3425 & \\
\hline
\end{array}$$

In practice, the work is as follows:

$$\begin{array}{r|l|l}
 & 11'90'25 & 345 \\
 & 9 & \\
\hline
64 & 290 & \\
 & 256 & \\
\hline
685 & 3425 & \\
 & 3425 & \\
\hline
\end{array}$$

Find the square root of—

3. 2304.
4. 3969.
5. 5184.
6. 7225.
7. 92416.
8. 15129.
9. 56644.
10. 321489.
11. 401956.
12. 11594025.
13. 11943936.
14. 6421156.
15. 5503716.
16. 642521104.
17. 412252416.
18. 14167696.
19. 250050969.
20. 1111374225.

If there is a remainder when the last period is used, the number has no exact square root, but the process may be continued by annexing periods of ciphers, and the approximate square root may thus be found to any degree of accuracy desired.

21. Find the square root of 2.

	2.	1.4142+, *Ans.*
	1	
24	100	
	96	
281	400	
	281	
2824	11900	
	11296	
28282	60400	
	56564	

Find the square root to four decimal places of—

22. 3.
23. 5.
24. 6.
25. 7.
26. 8.
27. 10.
28. .4.
29. .169.
30. .009216.
31. 3.1821.
32. 5.4218.
33. 11.3181.
34. 249.942.
35. .0007342.
36. 1.554433.
37. 92.536527.
38. 14561.836.
39. .01017946.

170. The square root of a fraction may be found by taking the square root of its terms, as follows:

EXAMPLES.

1. $\sqrt{\frac{16}{25}} = \frac{4}{5}$, *Ans.*

2. $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = \frac{1.73205+}{2} = .86602+$, *Ans.*

3. $\sqrt{\frac{2}{5}} = \sqrt{\frac{10}{25}} = \frac{3.16227+}{5} = .63245+$, *Ans.*

Find the square root to four decimal places of—

4. $\frac{121}{144}$.	7. $\frac{21}{64}$.	10. $\frac{5}{6}$.	13. $\frac{14}{15}$.
5. $\frac{484}{3969}$.	8. $\frac{2}{3}$.	11. $\frac{3}{11}$.	14. $\frac{9}{8}$.
6. $\frac{7}{49}$.	9. $\frac{7}{8}$.	12. $\frac{7}{10}$.	15. $25\frac{49}{64}$.

16. Find the value of $\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}}$.

CUBE ROOT OF POLYNOMIALS.

171.—1. Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

$$\begin{array}{l|l|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 & a + b, \textit{ Ans.} \\
 & a^3 & \\
\hline
\text{Trial divisor, } 3a^2 & 3a^2b + 3ab^2 + b^3 & \\
\text{1st correction, } \quad + 3ab & & \\
\text{2d correction, } \quad\quad + b^2 & & \\
\hline
\text{Complete divisor, } 3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 & \\
\hline
\end{array}$$

From the method of cubing a polynomial (Art. 160), the first term, a^3, is the cube of the first term of the root; hence, the first term of the root is the cube root of a^3, which is a. Subtracting its cube from the polynomial, we have $3a^2b + 3ab^2 + b^3$ remaining.

The first term of this remainder is equal to *three times the square of the first term into the second;* hence, we can find the second term of the root by dividing $3a^2b$ by $3a^2$, which gives b. The first correction is equal to three times the product of the terms, or $3ab$, and the second correction is equal to the square of the second term, or b^2.

Adding, we find the complete divisor equals $3a^2 + 3ab + b^2$. Multiplying by the second term and subtracting, there is no remainder. Hence, the cube root is $a + b$.

2. Find the cube root of $a^6 - 3a^5 + 5a^3 - 3a - 1$.

$$
\begin{array}{l|l|l}
 & a^6 - 3a^5 + 5a^3 - 3a - 1 & a^2 - a - 1, \textit{Ans.} \\
 & a^6 & \\
\hline
3a^4 & \quad -3a^5 + 5a^3 & \\
\quad -3a^3 & & \\
\qquad + a^2 & & \\
\cline{1-1}
3a^4 - 3a^3 + a^2 & & \\
\qquad a^2 & \quad -3a^5 + 3a^4 - a^3 & \\
\hline
3a^4 - 6a^3 + 3a^2 & \qquad -3a^4 + 6a^3 - 3a - 1 & \\
\qquad -3a^2 + 3a & & \\
\qquad\qquad +1 & & \\
\cline{1-1}
3a^4 - 6a^3 + 3a + 1 & \qquad -3a^4 + 6a^3 - 3a - 1 & \\
\hline
\end{array}
$$

The first two terms are found as in Ex. 1. The second *trial divisor* may be found by taking three times the square of the root already found—thus, $3(a^2 - a)^2 = 3a^4 - 6a^3 + 3a^2$; but a more convenient way is to repeat the second correction (a^2) and add it to the sum of the complete divisor and the two corrections, which gives $3a^4 - 6a^3 + 3a^2$.

Dividing the remainder by the new trial divisor, we have -1, the next term of the root. The first correction is $3(a^2 - a)(-1)$; the second correction is $(-1)^2$. Adding these corrections to the trial divisor, we have $3a^4 - 6a^3 + 3a + 1$. Multiplying by -1 and subtracting, there is no remainder. Hence, the cube root is $a^2 - a - 1$.

RULE.

Therefore, to extract the cube root of a polynomial,

Arrange the terms according to the powers of some letter.

Extract the cube root of the first term, write the result as the first term of the root, and subtract its cube from the polynomial.

Divide the first term of the remainder by three times the square of the root already found, and write the quotient for the next term of the root.

Add to the trial divisor three times the product of the first and second terms of the root, and the square of the second term. Multiply the complete divisor by the second term of the root, and subtract the product from the remainder.

To find the second trial divisor, repeat the second correction, and add it to the sum of the preceding complete divisor and the first and second corrections.

Continue in this manner until all the terms of the root are found.

EXAMPLES.

3. Find the cube root of $m^6 + 6m^5 - 40m^3 + 96m - 64$.

$$\begin{array}{l|l}
 & m^6 + 6m^5 - 40m^3 + 96m - 64 \;\big|\; m^2 + 2m - 4, \textit{ Ans.} \\
 & m^6 \\ \hline
3m^4 & 6m^5 - 40m^3 \\
\quad + 6m^3 & \\
\qquad + 4m^2 & \\ \hline
3m^4 + 6m^3 + 4m^2 & \\
\qquad 4m^2 & 6m^5 + 12m^4 + 8m^3 \\ \hline
3m^4 + 12m^3 + 12m^2 & - 12m^4 - 48m^3 + 96m - 64 \\
\quad - 12m^2 - 24m & \\
\qquad + 16 & \\ \hline
3m^4 + 12m^3 - 24m + 16 & - 12m^4 - 48m^3 + 96m - 64 \\ \hline
\end{array}$$

Find the cube root of the following:

4. $a^3 - 3a^2c + 3ac^2 - c^3$.
5. $1 - 6x + 12x^2 - 8x^3$.
6. $27m^3 - 54m^2n + 36mn^2 - 8n^3$.
7. $x^6 + 3x^5 - 5x^3 + 3x - 1$.
8. $a^6 - 9a^5 + 33a^4 - 63a^3 + 66a^2 - 36a + 8$.
9. $27m^6 + 171m^4 - 54m^5 - 188m^3 - 150m + 125 + 285m^2$.
10. $1 - 9x + 39x^2 - 99x^3 + 156x^4 - 144x^5 + 64x^6$.
11. $a^{6n} - 6a^{5n} + 21a^{4n} - 44a^{3n} + 63a^{2n} - 54a^n + 27$.
12. $a^3 - 3a^2 + 9a - 13 + \frac{18}{a} - \frac{12}{a^2} + \frac{8}{a^3}$.
13. $27m^6 - 54m^5 + 40m^3 - \frac{32}{3}m - \frac{64}{27}$.
14. $y^6 + 9y^5 - 135y^3 + 729y - 729$.
15. $x^9 - 3x^8 + 6x^7 - 10x^6 + 12x^5 - 12x^4 + 10x^3 - 6x^2 + 3x - 1$.

CUBE ROOT OF NUMBERS.

172. Algebra also furnishes a very satisfactory explanation of the **Cube Root of Numbers.**

PRINCIPLE I.

The cube of a number consists of three times as many figures as the number, or three times as many less one or two.

$$1^3 = 1. \qquad 10^3 = 1000.$$
$$3^3 = 27. \qquad 30^3 = 27000.$$
$$9^3 = 729. \qquad 99^3 = 970299.$$

The cube of 1, the smallest number of one figure, is 1; the cube of 3 is 27, and the cube of 9, the largest number of one figure, is 729; hence, the cube of a number of one figure contains one, two, or three figures; that is, *three times as many, or three times as many less one or two.*

In the same way we find the principle true for numbers of two figures; hence, we infer that it is generally true.

PRINCIPLE II.

If a number be pointed off into periods of three figures each, beginning at the right, the number of full periods, together with the partial period on the left if there be one, will equal the number of figures in the root. This is evident from Prin. 1.

PRINCIPLE III.

The cube of a decimal contains three times as many figures as the decimal itself.

$$.1^3 = .001. \qquad .01^3 = .000001.$$
$$.9^3 = .729. \qquad .99^3 = .970299.$$

It is thus seen that the cube of a decimal of one figure contains three figures, and the cube of a decimal of two figures contains six figures; hence, we infer that the principle is generally true.

PRINCIPLE IV.

If we represent the units by ***u****, the tens by* ***t****, and the hundreds by* ***h****, we will have the following formulas:*

$$(t + u)^3 = t^3 + 3t^2u + 3tu^2 + u^3,$$
$$(h + t + u)^3 = h^3 + 3h^2t + 3ht^2 + t^3 + 3(h + t)^2u + 3(h + t)u^2 + u^3.$$

1. Find the cube root of 300763.

$$\begin{array}{lr|r|l}
t^3 + 3t^2u + 3tu^2 + u^3 = & & 300'763 & 60 \\
t^3 = \qquad 60^3 = & & 216\,000 & \underline{\;7} \\ \hline
(3t^2 + 3tu + u^2)u \quad = & & 84\,763 & 67, \textit{Ans.} \\
3t^2 = \;\; 3 \times 60^2 \quad = & 10800 & & \\
+\,3tu = 3 \times 60 \times 7 = & 1260 & & \\
+\,u^2 = \qquad 7^2 = & 49 & & \\ \hline
\multicolumn{3}{l}{(3t^2 + 3tu + u^2)u = 12109 \times 7 = 84763} & \\ \hline
\end{array}$$

Separating the number into periods of three figures, we find the root will contain two figures (Prin. 2); hence, the root will consist of tens and units, while the number consists of $t^3 + 3t^2u + 3tu^2 + u^3$.

The greatest number of *tens* whose cube is contained in 300763 is 6 *tens*, or 60. Subtracting the cube of 60 from 300763, we have 84763 remaining, which equals $3t^2u + 3tu^2 + u^3$.

Since $3t^2u$ is always greater than $3tu^2 + u^3$, 84763 consists principally of $3t^2u$; hence, we can find the units by dividing 84763 by $3t^2$, or 10800. Dividing 84763 by 10800 gives 7, the unit's figure. The first correction is $3tu = 3 \times 60 \times 7 = 1260$, and the second correction is $u^2 = 7^2 = 49$; taking the sum, we have 12109. Multiplying by 7 and subtracting, there is no remainder. Hence, the cube root is 60 + 7, or 67.

In practice we *omit the ciphers*, and the root is written simply 67.

2. Find the cube root of 95256152263.

$$\begin{array}{r|l|l}
 & 95'256'152'263 & 4567, \textit{Ans.} \\
 & 64 & \\ \hline
48 & 31\,256 & \\
60 & & \\
25 & & \\ \hline
5425 & & \\
(25) & 27125 & \\ \hline
6075 & 4131152 & \\
810 & & \\
36 & & \\ \hline
615636 & & \\
(36) & 3693816 & \\ \hline
623808 & 437336263 & \\
9576 & & \\
49 & & \\ \hline
62476609 & 437336263 & \\ \hline
\end{array}$$

The first two figures of the root are found as in Ex. 1. The second trial divisor, 6075, may be found by taking three times 45^2; but a more convenient way is to repeat the second correction (25), and add it to the sum of the complete divisor and the two corrections, which gives 6075.

Dividing 4131152 by 6075, omitting the last two figures, we find the next figure of the root to be 6. Adding the first correction, $3 \times 6 \times 45 = 810$, and the second correction, $6^2 = 36$, we have the complete divisor 615636. Multiplying by 6 and subtracting, we have 437336263.

The third trial divisor is found by repeating the second correction (36), and adding it to the last complete divisor and the first and second corrections, which gives 623808. Dividing 437336263 by 623808, omitting the last two figures, we find the next figure of the root to be 7. Adding the first and second corrections, multiplying the sum by 7, and subtracting, we have no remainder. Hence, the cube root is 4567.

Therefore, to extract the cube root of a number,

RULE.

Separate the number into periods of three figures each, beginning with the unit's place.

Find the largest number whose cube is contained in the left-hand period, and write it as the first figure of the root; subtract its cube from the period, and bring down the next period for a new dividend.

Divide the new dividend, omitting the last two figures, by three times the square of the root already found, and write the quotient as the second figure of the root.

To the trial divisor add three times the product of the last figure of the root and the part of the root previously found, written one figure to the right, and the square of the last figure of the root, written two figures to the right, for the complete divisor.

Multiply the complete divisor by the figure of the root last found, and subtract the product from the remainder.

Thus continue until all the periods are used.

EXAMPLES.

Find the cube root of—

3. 262144.
4. 571787.
5. 778688.
6. 2097152.
7. 41063625.
8. 5545233.

9. 517781627.
10. 130323843.
11. 60236288.
12. 1371330631.
13. 277167808.
14. 1879080904.
15. 80677568161.
16. 706157817625.
17. 12905114402816.

If there is a remainder when the last period is used, the number has no exact cube root, but the process may be continued by annexing periods of ciphers, and the approximate cube root can be found to any degree of accuracy desired.

18. Find the cube root of 4.

	4.000	1.5873+, *Ans.*
	1	
3	3000	
15		
25		
475		
25	2375	
675	625000	
360		
64		
71164		
64	569312	
74892	55688000	
3318		
49		
7522429		
49	52657003	
7555707	3030997000	
14283		
9		
755713539	2267140617	

Find the cube root to four decimal places of—

19. 1.728.
20. 15.625.
21. 2.
22. 10.01.
23. 756.0657.
24. 3.
25. 5.
26. 6.
27. .0081.
28. 64.345.
29. .27.
30. .09.
31. 6.4.
32. 189.5.
33. 5981.072.

173. The cube root of a fraction may be found by taking the cube root of its terms.

The following method is preferred:

$$1.\ \sqrt[3]{\frac{64}{125}} = \frac{4}{5},\ \textit{Ans.}$$

$$2.\ \sqrt[3]{\frac{5}{8}} = \frac{\sqrt[3]{5}}{2} = \frac{1.7099+}{2} = .8549+,\ \textit{Ans.}$$

$$3.\ \sqrt[3]{\frac{3}{4}} = \sqrt[3]{\frac{6}{8}} = \frac{\sqrt[3]{6}}{2} = \frac{1.8171+}{2} = .9085+,\ \textit{Ans.}$$

Find the cube root to four decimal places of—

4. $\frac{1728}{15625}$.
5. $\frac{24389}{405224}$.
6. $\frac{42875}{54872}$.
7. $\frac{123456}{654321}$.
8. $\frac{3}{8}$.
9. $\frac{7}{27}$.
10. $\frac{15}{64}$.
11. $\frac{521}{125}$.
12. $\frac{2}{3}$.
13. $\frac{6}{25}$.
14. $\frac{8}{9}$.
15. $\frac{41}{58}$.
16. $64\frac{27}{125}$.
17. $34\frac{21}{64}$.
18. $3\frac{1}{2}$.
19. $17\frac{17}{71}$.

THEORY OF EXPONENTS.

174. Thus far we have considered only *positive integral exponents;* but the generalizations of algebra require the use of *fractional* and of *negative* exponents. We will therefore proceed to investigate their meaning and use.

175. Positive integral exponents.

By ART. 6, if n is a positive integer, a^n equals the product obtained by using a as a factor n times.

Thus, $a \times a \times a$ ..... to n factors equals a^n.

By this definition, $a^m = a \cdot a \cdot a$ ..... to m factors.

And $a^n = a \cdot a \cdot a$ ..... to n factors.

Hence:

PRINCIPLE I.

$$a^m \times a^n = a \cdot a \cdot a \text{ ..... to } (m+n) \text{ factors;}$$

$$= a^{m+n} \text{ (ART. 71, Prin. 3).}$$

PRINCIPLE II.

$$\text{Also, } a^m \div a^n = \frac{a \cdot a \cdot a \,.\,.\,.\,.\,.\, \text{ to } m \text{ factors}}{a \cdot a \cdot a \,.\,.\,.\,.\,.\, \text{ to } n \text{ factors}};$$

$$= a \cdot a \cdot a \,.\,.\,.\,.\,.\, \text{ to } (m - n) \text{ factors};$$

$$= a^{m-n} \text{ (Art. 80, Prin. 2).}$$

PRINCIPLE III.

$$\text{Also, } (a^m)^n = a^m a^m a^m \,.\,.\,.\,.\,.\, \text{ to } n \text{ factors};$$

$$= a^{m+m+m \,.\,.\,.\,.\,.\, \text{to } n \text{ terms}}; \text{ (Art. 71, Prin. 3).}$$

$$= a^{nm}.$$

PRINCIPLE IV.

$$\text{Also, } \sqrt[n]{a^{mn}} = \sqrt[n]{a^m a^m a^m \,.\,.\,.\,.\,.\, \text{ to } n \text{ factors}};$$

$$= a^m \text{ (Art. 8).}$$

These principles will now be proved universally true.

176. Fractional exponents.

A *fractional exponent* arises from evolution by dividing the exponent of the power by the index of the root, when the former is not a multiple of the latter.

Since by the definition of a^n the exponent n means the number of times a is taken as a factor, such expressions as

$$a^{\frac{4}{3}} \text{ and } a^{-n}$$

have no meaning whatever, because we cannot write a $\frac{4}{3}$ times or $-n$ times. We must therefore either change the definition of an exponent, or put such a meaning into a fractional or a negative exponent as will harmonize with the definition; for it is very important that algebraic symbols always conform to the same laws. Let us therefore assume the definition of Art. 6 to be true, and see what meaning must be given to a fractional or a negative exponent, so that $a^m \times a^n = a^{m+n}$ may always be true.

What is the meaning of $a^{\frac{4}{3}}$?

Since Prin. 1, Art. 175, is to hold universally, we must have

$$a^{\frac{4}{3}} \times a^{\frac{4}{3}} \times a^{\frac{4}{3}} = a^{\frac{4}{3}+\frac{4}{3}+\frac{4}{3}} = a^4.$$

That is, $a^{\frac{4}{3}}$ must be such a quantity that, when cubed, will produce a^4. It is therefore evidently the cube root of a^4; or $a^{\frac{4}{3}} = \sqrt[3]{a^4}$.

What is the meaning of $a^{\frac{m}{n}}$?

By Prin. 1, Art. 175, we have

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \ldots\ldots \text{ to } n \text{ factors}$$

$$= a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}\ldots\ldots \text{to } n \text{ terms}}; \quad (\text{Art. } 71, \text{ Prin. } 3).$$

$$= a^{\frac{m}{n}\times n} = a^m.$$

That is, $a^{\frac{m}{n}}$ must be such a quantity that, when raised to the nth power, it will produce a^m. It is therefore evidently the nth root of a^m, or $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

Hence, in a fractional exponent,

The numerator denotes a power, and the denominator a root.

EXAMPLES.

Express the following with radical signs:

1. $a^{\frac{1}{2}}$.
2. $x^{\frac{2}{3}}$.
3. $a^{\frac{5}{6}}$.
4. $3a^{\frac{1}{3}}$.
5. $ax^{\frac{1}{n}}$.
6. $a^{\frac{1}{2}}b^{\frac{1}{3}}$.
7. $a^{\frac{1}{n}}b^{\frac{1}{n}}$.
8. $a^{\frac{1}{2}}x^{\frac{1}{n}}$.
9. $6m^{\frac{x}{2}}$.
10. $4x^{\frac{2}{3}}y^{\frac{5}{6}}$.
11. $2m^{\frac{1}{2}}n^{\frac{1}{3}}$.
12. $4x^{\frac{2}{3}}y^{\frac{3}{4}}$.
13. $(3a)^{\frac{2}{3}}$.
14. $3a^{\frac{2}{3}}$.
15. $a^{\frac{1}{2}}x^{\frac{1}{5}}$.
16. $x^{\frac{2}{3}}y^{\frac{3}{4}}z^{\frac{1}{6}}$.

Express the following with fractional exponents:

17. $\sqrt{x}$. 21. $\sqrt[4]{a^n}$. 25. $\sqrt[n]{x^2y^4}$. 29. $\sqrt[n]{ab^2c^3}$.

18. $\sqrt[3]{a}$. 22. $5\sqrt[3]{3}$. 26. $\sqrt[6]{a^2y^3}$. 30. $\sqrt{a^nb^mc}$.

19. $\sqrt[n]{m}$. 23. $a\sqrt[n]{x^m}$. 27. $\sqrt[12]{x^3y^4z^6}$. 31. $\sqrt[3]{a^2b^6}$.

20. $\sqrt[3]{a^2}$. 24. $\sqrt[n]{a^2b^3}$. 28. $\sqrt[n]{a^{2n}y^3}$. 32. $m\sqrt[n]{m^{2n}b}$.

Find the values of the following:

33. $4^{\frac{1}{2}}$. 37. $16^{\frac{3}{2}}$. 41. $729^{\frac{4}{3}}$. 45. $(\frac{4}{81})^{\frac{1}{2}}$.

34. $8^{\frac{1}{3}}$. 38. $4^{\frac{5}{2}}$. 42. $64^{\frac{5}{6}}$. 46. $(\frac{25}{49})^{\frac{3}{2}}$.

35. $16^{\frac{1}{4}}$. 39. $36^{\frac{3}{2}}$. 43. $(-32)^{\frac{3}{5}}$. 47. $(\frac{64}{729})^{\frac{5}{6}}$.

36. $(-27)^{\frac{1}{3}}$. 40. $(-8)^{\frac{2}{3}}$. 44. $(-125)^{\frac{2}{3}}$. 48. $(-\frac{216}{343})^{\frac{1}{3}}$.

177. Zero exponent.

A *zero exponent* arises from dividing a quantity with any exponent by the same quantity with the same exponent.

Thus, $a^5 \div a^5 = a^{5-5} = a^0$.

What is the meaning of a^0?

Since Prin. 1, ART. 175, is to hold universally, we must have

$$a^n \times a^0 = a^{n+0} = a^n.$$

$$\therefore a^0 = a^n \div a^n = 1.$$

Hence, *any quantity whose exponent is* 0 *equals* 1.

178. Negative exponents.

A *negative exponent* arises by dividing a quantity with any exponent by the same quantity with a larger exponent.

Thus, $a^4 \div a^7 = a^{4-7} = a^{-3}$.

What is the meaning of a^{-n}?

Since Prin. 1, Art. 175, is to hold universally, we must have

$$a^n \times a^{-n} = a^0 = 1.$$

$$a^{-n} = \frac{1}{a^n}.$$

CUBE ROOT OF NUMBERS.

172. Algebra also furnishes a very satisfactory explanation of the **Cube Root of Numbers.**

PRINCIPLE I.

The cube of a number consists of three times as many figures as the number, or three times as many less one or two.

$1^3 = 1.$ $10^3 = 1000.$

$3^3 = 27.$ $30^3 = 27000.$

$9^3 = 729.$ $99^3 = 970299.$

The cube of 1, the smallest number of one figure, is 1; the cube of 3 is 27, and the cube of 9, the largest number of one figure, is 729; hence, the cube of a number of one figure contains one, two, or three figures; that is, *three times as many, or three times as many less one or two.*

In the same way we find the principle true for numbers of two figures; hence, we infer that it is generally true.

PRINCIPLE II.

If a number be pointed off into periods of three figures each, beginning at the right, the number of full periods, together with the partial period on the left if there be one, will equal the number of figures in the root. This is evident from Prin. 1.

PRINCIPLE III.

The cube of a decimal contains three times as many figures as the decimal itself.

$.1^3 = .001.$ $.01^3 = .000001.$

$.9^3 = .729.$ $.99^3 = .970299.$

It is thus seen that the cube of a decimal of one figure contains three figures, and the cube of a decimal of two figures contains six figures; hence, we infer that the principle is generally true.

PRINCIPLE IV.

*If we represent the units by **u**, the tens by **t**, and the hundreds by **h**, we will have the following formulas:*

$$(t + u)^3 = t^3 + 3t^2u + 3tu^2 + u^3,$$

$$(h + t + u)^3 = h^3 + 3h^2t + 3ht^2 + t^3 + 3(h + t)^2u + 3(h + t)u^2 + u^3.$$

1. Find the cube root of 300763.

$t^3 + 3t^2u + 3tu^2 + u^3 =$	300'763	60
$t^3 = \quad 60^3 =$	216 000	7
$(3t^2 + 3tu + u^2)u \quad =$	84 763	67, *Ans.*
$3t^2 = 3 \times 60^2 = 10800$		
$+ 3tu = 3 \times 60 \times 7 = 1260$		
$+ u^2 = 7^2 = 49$		
$(3t^2 + 3tu + u^2)u = 12109 \times 7 = 84763$		

Separating the number into periods of three figures, we find the root will contain two figures (Prin. 2); hence, the root will consist of tens and units, while the number consists of $t^3 + 3t^2u + 3tu^2 + u^3$.

The greatest number of *tens* whose cube is contained in 300763 is 6 *tens*, or 60. Subtracting the cube of 60 from 300763, we have 84763 remaining, which equals $3t^2u + 3tu^2 + u^3$.

Since $3t^2u$ is always greater than $3tu^2 + u^3$, 84763 consists principally of $3t^2u$; hence, we can find the units by dividing 84763 by $3t^2$, or 10800. Dividing 84763 by 10800 gives 7, the unit's figure. The first correction is $3tu = 3 \times 60 \times 7 = 1260$, and the second correction is $u^2 = 7^2 = 49$; taking the sum, we have 12109. Multiplying by 7 and subtracting, there is no remainder. Hence, the cube root is 60 + 7, or 67.

In practice we *omit the ciphers*, and the root is written simply 67.

2. Find the cube root of 95256152263.

	95'256'152'263	4567, *Ans.*
	64	
48	31 256	
60		
25		
5425		
(25)	27125	
6075	4131152	
810		
36		
615636		
(36)	3693816	
623808	437336263	
9576		
49		
62476609	437336263	

17. Multiply $m^{\frac{2}{3}} + m^{\frac{1}{3}}n^{\frac{1}{3}} + n^{\frac{2}{3}}$ by $m^{\frac{1}{3}} - n^{\frac{1}{3}}$.

$$\begin{array}{llll}
m^{\frac{2}{3}} + m^{\frac{1}{3}}n^{\frac{1}{3}} + n^{\frac{2}{3}} & & & \\
\underline{m^{\frac{1}{3}} - n^{\frac{1}{3}}\qquad\qquad} & & & \\
m & + m^{\frac{2}{3}}n^{\frac{1}{3}} & + m^{\frac{1}{3}}n^{\frac{2}{3}} & \\
\underline{} & \underline{- m^{\frac{2}{3}}n^{\frac{1}{3}}} & \underline{- m^{\frac{1}{3}}n^{\frac{2}{3}}} & \underline{- n} \\
m & & & - n, \textit{ Ans.}
\end{array}$$

Multiply the following:

18. $a^{\frac{1}{2}} + b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.

19. $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.

20. $x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$.

21. $a^{\frac{8}{3}} + a^{\frac{4}{3}}b^{\frac{4}{3}} + b^{\frac{8}{3}}$ by $a^{\frac{4}{3}} - b^{\frac{4}{3}}$.

22. $a^{\frac{4}{3}} - a^{\frac{2}{3}}x^{\frac{2}{3}} + x^{\frac{4}{3}}$ by $a^{\frac{2}{3}} + x^{\frac{2}{3}}$.

23. $a^{-4} - a^{-2}x^{-4} + x^{-8}$ by $a^{-2} + x^{-4}$.

24. $x^{\frac{3}{4}} + x^{\frac{1}{2}}y^{-\frac{1}{4}} + x^{\frac{1}{4}}y^{-\frac{1}{2}} + y^{-\frac{3}{4}}$ by $x^{\frac{1}{4}} - y^{-\frac{1}{4}}$.

25. $m^4 + m^2 + 1$ by $m^{-4} - m^{-2} + 1$.

26. $(x^{\frac{3}{4}} - y^{\frac{3}{4}})(x^{\frac{3}{4}} + y^{\frac{3}{4}})(x^{\frac{3}{2}} + y^{\frac{3}{2}})$.

27. $(a^{\frac{1}{8}} - b^{\frac{1}{8}})(a^{\frac{1}{8}} + b^{\frac{1}{8}})(a^{\frac{1}{4}} + b^{\frac{1}{4}})(a^{\frac{1}{2}} + b^{\frac{1}{2}})$.

182. To prove that $a^m \div a^n = a^{m-n}$ is universally true.

We have proved $a^m \times a^{-n} = a^{m-n}$ (ART. 181).

But $\qquad a^m \times a^{-n} = \dfrac{a^m}{a^n} = a^m \div a^n = a^{m-n}$.

Hence, *the exponential law in division is universally true, whether m and n be positive or negative, integral or fractional.*

$$\text{Then } a^4 \div a^{-2} = a^{4+2} = a^6.$$

$$a^{-\frac{1}{2}} \div a^{-\frac{2}{3}} = a^{-\frac{1}{2}+\frac{2}{3}} = a^{\frac{1}{6}}, \text{ etc.}$$

EXAMPLES.

Divide the following:

1. $a^5 \div a^4$.
2. $a^3 \div a^5$.
3. $a^3 \div a^{-2}$.
4. $a^4 \div a^{-1}$.
5. $a^{-3} \div a^{-2}$.
6. $a^{\frac{1}{2}} \div a^{\frac{1}{3}}$.
7. $a^{\frac{2}{3}} \div a^{-\frac{3}{4}}$.
8. $a^{-2} \div a^{\frac{1}{2}}$.
9. $x^{-\frac{1}{5}} \div x^{-1}$.
10. $m^{-\frac{1}{2}} \div m^{-\frac{1}{3}}$.
11. $y^n \div y^{-m}$.
12. $x^{-n} \div x^m$.
13. $z^{-n} \div z^{-m}$.
14. $a^{\frac{1}{n}} \div a^{\frac{1}{m}}$.
15. $b^{-n} \div b^{\frac{1}{n}}$.

16. Divide $m + n$ by $m^{\frac{1}{3}} + n^{\frac{1}{3}}$.

$$\begin{array}{r|l}
m + n & m^{\frac{1}{3}} + n^{\frac{1}{3}} \\ \cline{2-2}
\underline{m + m^{\frac{2}{3}}n^{\frac{1}{3}}} & \underline{m^{\frac{2}{3}}} - m^{\frac{1}{3}}n^{\frac{1}{3}} + n^{\frac{2}{3}},\ \textit{Ans.}
\end{array}$$

$$\begin{array}{l}
-m^{\frac{2}{3}}n^{\frac{1}{3}} + n \\
\underline{-m^{\frac{2}{3}}n^{\frac{1}{3}} - m^{\frac{1}{3}}n^{\frac{2}{3}}\qquad\qquad} \\
\qquad\qquad m^{\frac{1}{3}}n^{\frac{2}{3}} + n. \\
\qquad\qquad \underline{m^{\frac{1}{3}}n^{\frac{2}{3}} + n.}
\end{array}$$

Divide—

17. $a - b$ by $a^{\frac{1}{2}} + b^{\frac{1}{2}}$.
18. $x - y$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.
19. $x + y$ by $x^{\frac{1}{5}} + y^{\frac{1}{5}}$.
20. $a^2 + b^2$ by $a^{\frac{2}{3}} + b^{\frac{2}{3}}$.
21. $a^4 + b^4$ by $a^{\frac{4}{3}} + b^{\frac{4}{3}}$.
22. $a^{-3n} - b^{-3n}$ by $a^{-n} - b^{-n}$.
23. $x^{-6} + b^{-12}$ by $x^{-2} + b^{-4}$.
24. $m^{\frac{4}{3}} - 2m^{\frac{2}{3}}n^{\frac{2}{3}} + n^{\frac{4}{3}}$ by $m^{\frac{2}{3}} - n^{\frac{2}{3}}$.
25. $m^4 + 1 + m^{-4}$ by $m^{-4} - m^{-2} + 1$.
26. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - a^{\frac{1}{6}}b^{\frac{1}{6}} + b^{\frac{1}{3}}$.
27. $a + 3a^{\frac{2}{3}}b^{-\frac{1}{3}} + 3a^{\frac{1}{3}}b^{-\frac{2}{3}} + b^{-1}$ by $a^{\frac{1}{3}} + b^{-\frac{1}{3}}$.

183. To prove that $(a^m)^n = a^{nm}$ is universally true.

1. Let n be a positive fraction, as $\frac{p}{q}$.

$$\text{Then, } (a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}} = a^{\frac{mp}{q}} = a^{m \times \frac{p}{q}}.$$

2. Let n be negative, as $-p$.

$$\text{Then, } (a^m)^{-p} = \frac{1}{(a^m)^p} = \frac{1}{a^{mp}} = a^{-mp} = a^{m \times (-p)}.$$

Hence, *the law relating to exponents in involution is universally true, whether m and n be positive or negative, integral or fractional.*

$$\text{Then, } (a^3)^{-2} = a^{-6};$$

$$(a^{-\frac{1}{3}})^3 = a^{-1};$$

$$\left(a^{-\frac{2}{3}}\right)^{-\frac{3}{2}} = a, \text{ etc.}$$

EXAMPLES.

Find the values of—

1. $(a^2)^{-3}$.
2. $(a^{-2})^{\frac{1}{2}}$.
3. $(a^{-3})^{-2}$
4. $(a^{\frac{1}{2}})^{\frac{5}{2}}$.
5. $(m^{-\frac{2}{3}})^{-2}$.
6. $(n^{-1})^{-n}$.
7. $(\sqrt{a})^{-2}$.
8. $(x^{-\frac{1}{2}})^4$.
9. $\left\{(a^{-2})^2\right\}^{-3}$.
10. $\left\{(a^{-\frac{1}{2}})^{-2}\right\}^{-5}$.
11. $\left\{(m^{-2})^{-\frac{1}{2}}\right\}^4$.
12. $\left\{(x^{-2})^{-3}\right\}^{\frac{1}{2}}$.

MISCELLANEOUS EXAMPLES.

184. Multiply—

1. $x^{\frac{n}{2}} + y^{\frac{n}{2}}$ by $x^{\frac{n}{2}} - y^{\frac{n}{2}}$.
2. $m^{-\frac{2}{3}} + m^{-\frac{1}{3}} + 1$ by $m^{-\frac{1}{3}} - 1$.
3. $a^{\frac{16}{5}} - a^{\frac{12}{5}} + a^{\frac{8}{5}} - a^{\frac{4}{5}} + 1$ by $a^{\frac{4}{5}} + 1$.
4. $a^{\frac{1}{2}} + a^{\frac{1}{3}}b^{\frac{1}{6}} + a^{\frac{1}{6}}b^{\frac{1}{3}} + b^{\frac{1}{2}}$ by $a^{\frac{1}{6}} - b^{\frac{1}{6}}$.
5. $x^{\frac{4}{5}} + x^{\frac{2}{5}}y^{-\frac{2}{5}} + y^{-\frac{4}{5}}$ by $x^{\frac{4}{5}} - x^{\frac{2}{5}}y^{-\frac{2}{5}} + y^{-\frac{4}{5}}$.

Divide—

6. $a^n - b^n$ by $a^{\frac{n}{3}} - b^{\frac{n}{3}}$. 8. $m^{\frac{4x}{3}} - n^{\frac{4y}{3}}$ by $m^{\frac{x}{3}} + n^{\frac{y}{3}}$.

7. $a^{\frac{3}{5}} + b^{\frac{3}{5}}$ by $a^{\frac{1}{5}} + b^{\frac{1}{5}}$. 9. $a + b$ by $a^{\frac{1}{7}} + b^{\frac{1}{7}}$.

10. $a^{-6} + a^{-3}y^{-3} + y^{-6}$ by $a^{-3} + a^{-\frac{3}{2}}y^{-\frac{3}{2}} + y^{-3}$.

Expand—

11. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$. 14. $(a^{\frac{1}{2}} - b)^3$.

12. $(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2$. 15. $(a^{\frac{1}{2}} - c^{\frac{1}{2}})^4$.

13. $(a^{\frac{1}{3}} + b^{\frac{1}{3}})^3$. 16. $(a^{\frac{n}{3}} + b^{\frac{m}{2}})(a^{\frac{n}{3}} - b^{\frac{m}{2}})$.

Resolve into two binomial factors—

17. $m - n$. 19. $x^{\frac{1}{3}} - b^{\frac{1}{3}}$. 21. $x^{-2} - b^{-2}$.

18. $m^{-1} - n^{-1}$. 20. $a^3 - b^3$. 22. $m^{-x} - n^{-y}$.

Resolve into two trinomial factors—

23. $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b$. 25. $a^{\frac{4}{3}} + a^{\frac{2}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$.

24. $x^{-4} + x^{-2}y^{-2} + y^{-4}$. 26. $x^{-6n} + x^{-3n}y^{3n} + y^{6n}$.

Extract the square root of—

27. $x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$. 28. $9a^{\frac{2}{5}} - 12a^{\frac{1}{5}}c^{\frac{1}{5}} + 4c^{\frac{2}{5}}$.

29. $a^{\frac{4}{3}} + 4ab^{\frac{1}{3}} + 10a^{\frac{2}{3}}b^{\frac{2}{3}} + 12a^{\frac{1}{3}}b + 9b^{\frac{4}{3}}$.

Extract the cube root of—

30. $27m - 54m^{\frac{2}{3}}n^{\frac{1}{3}} + 36m^{\frac{1}{3}}n^{\frac{2}{3}} - 8n$.

31. $a^2 + 3a^{\frac{5}{3}} - 5a + 3a^{\frac{1}{3}} - 1$.

32. $a - 3a^{\frac{2}{3}} + 9a^{\frac{1}{3}} - 13 + 18a^{-\frac{1}{3}} - 12a^{-\frac{2}{3}} + 8a^{-1}$.

RADICALS.

185. A **Radical** is an indicated root of an imperfect power; as, $\sqrt{a}$, $\sqrt[3]{a+b}$, etc.

A *Rational Quantity* is one which can be expressed without the radical sign or fractional exponent; as, a, $\sqrt[3]{8}$, etc. Rational quantities can always be expressed in the radical form.

Thus, $2 = \sqrt{4}$, $a = \sqrt{a^2}$, etc.

Radicals are also called *Irrational Quantities*, or *Surds.*

186. The **Degree** of a radical is denoted by the index of the radical sign or by the denominator of the fractional exponent.

Thus, $\sqrt{a}$ and $a^{\frac{3}{2}}$ are radicals of the second degree.

$\sqrt[3]{a}$ and $a^{\frac{2}{3}}$ are radicals of the third degree.

$\sqrt[n]{a}$ and $a^{\frac{m}{n}}$ are radicals of the nth degree.

187. **Similar Radicals** are those which have a common index, and have the same quantity under the radical sign; as, $a\sqrt[3]{b}$ and $-c\sqrt[3]{b}$.

188. The **Coefficient** of a radical is the quantity or factor prefixed to it.

Thus, in $4\sqrt{a}$, 4 is the coefficient.

REDUCTION OF RADICALS.

189. The reduction of radicals depends upon the following important

PRINCIPLE.

Any root of the product of two or more quantities is equal to the product of the same roots of those quantities.

DEMONSTRATION.—$\sqrt[n]{ab} = a^{\frac{1}{n}}b^{\frac{1}{n}}$ (ART. 167).

But $a^{\frac{1}{n}}b^{\frac{1}{n}} = \sqrt[n]{a} \times \sqrt[n]{b}$.

190. To reduce a radical to its simplest form.

A radical is in its *simplest form* when the quantity under the radical sign contains no factor of which the given root can be taken; when it is integral; or when it cannot be made a perfect power of the degree denoted by a factor of the index.

Thus, $\sqrt{12}$, $\sqrt[3]{\frac{1}{2}}$, and $\sqrt[6]{8}$ are *not* in their simplest forms.

CASE I.

191. When the quantity under the radical sign contains a factor of which the given root can be taken.

1. Reduce $\sqrt{12}$ to its simplest form.

$$\sqrt{12}=\sqrt{4\times 3}=\sqrt{4}\times\sqrt{3}\text{ (Art. 189)}=2\sqrt{3},\ \textit{Ans.}$$

2. Reduce $\sqrt[3]{27a^5b^3-27a^3b^5}$ to its simplest form.

$$\sqrt[3]{27a^5b^3-27a^3b^5}=\sqrt[3]{27a^3b^3(a^2-b^2)}=\sqrt[3]{27a^3b^3}\times\sqrt[3]{a^2-b^2}\text{ (Art. 189)}$$
$$=3ab\sqrt[3]{a^2-b^2},\ \textit{Ans.}$$

Hence the

RULE.

Resolve the quantity under the radical sign into two factors, one of which is a perfect power of the given root; extract the required root of this factor, and prefix it as a coefficient to the indicated root of the other factor.

EXAMPLES.

Reduce the following to their simplest forms:

3. $\sqrt{72}$.
4. $\sqrt{75}$.
5. $\sqrt{108}$.
6. $\sqrt[3]{54}$.
7. $\sqrt[3]{128}$.
8. $\sqrt[3]{-250}$.
9. $2\sqrt{8a^3b^2}$.
10. $3\sqrt{98m^3n^4}$.
11. $6\sqrt[3]{81x^5y^8}$.
12. $\sqrt{100a^5b^6}$.
13. $2\sqrt{x^3-x^2y}$.
14. $\sqrt{x^3(x+y)^2}$.
15. $\sqrt[3]{16x^4-24x^7}$.
16. $\sqrt[4]{(x+y)^4x^5}$.
17. $\sqrt[3]{3b^3+4b^4c}$.

18. $\sqrt{4a^3 + 8a^2b + 4ab^2}$.

19. $\sqrt{(a^2 - b^2)(a - b)}$.

20. $\sqrt{(a^2 + b^2)(a^4 - b^4)}$.

21. $\sqrt[3]{(a^3 + b^3)(a + b)^2}$.

22. $\sqrt[3]{(x - x^{-1})^2 (x^3 - x^{-3})}$.

23. $\sqrt{(x^2 + y^2)(x^6 + y^6)}$.

24. $(m + n)\sqrt{m^2n - 2mn^2 + n^3}$.

25. $\sqrt[3]{2a^4 - 6a^3b + 6a^2b^2 - 2ab^3}$.

26. $\sqrt[3]{3a^3 + 27a^2b + 81ab^2 + 81b^3}$.

CASE II.

192. When the quantity under the radical sign is fractional.

1. Reduce $\sqrt[3]{\frac{1}{2}}$ to its simplest form.

$$\sqrt[3]{\frac{1}{2}} = \sqrt[3]{\frac{1 \times 4}{2 \times 4}} = \sqrt[3]{\frac{1}{8} \times 4} = \sqrt[3]{\frac{1}{8}} \times \sqrt[3]{4} = \frac{1}{2}\sqrt[3]{4},\ \textit{Ans.}$$

2. Reduce $\sqrt{\frac{3n}{5m}}$ to its simplest form.

$$\sqrt{\frac{3n}{5m}} = \sqrt{\frac{3n \times 5m}{5m \times 5m}} = \sqrt{\frac{1}{5^2m^2} \times 15mn} = \sqrt{\frac{1}{5^2m^2}} \times \sqrt{15mn}$$
$$= \frac{1}{5m}\sqrt{15mn},\ \textit{Ans.}$$

Hence the

RULE.

Multiply both terms of the fraction by such a quantity as will make the denominator a perfect power of the required root; then proceed as before.

EXAMPLES.

Reduce the following to their simplest forms:

3. $\sqrt{\frac{1}{2}}$.

4. $\sqrt{\frac{1}{3}}$.

5. $\sqrt[3]{\frac{3}{4}}$.

6. $\sqrt[3]{\frac{5}{9}}$.

7. $\sqrt[3]{\frac{7x}{25}}$.

8. $a\sqrt{\frac{4x}{3a}}$.

9. $\sqrt[4]{\frac{3x}{8}}$.
11. $a\sqrt[n]{\frac{a^2}{b}}$.
13. $4\sqrt{\frac{8x^3y}{3n^2}}$.

10. $\frac{2}{3}\sqrt[3]{\frac{2}{3}}$.
12. $\frac{a}{b}\sqrt[3]{\frac{b^4}{a}}$.
14. $\sqrt[3]{\frac{xy}{a^2b^n}}$.

15. $(m-n)\sqrt{\frac{m-n}{m+n}}$.
17. $(3+x)\sqrt{\frac{3-x}{3+x}}$.

16. $\frac{1}{x+y}\sqrt{\frac{x+y}{x-y}}$.
18. $(x+y)\sqrt[3]{\frac{x-y}{(x+y)^2}}$.

CASE III.

193. When the quantity under the radical sign can be made a perfect power of the degree denoted by a factor of the index.

1. Reduce $\sqrt[12]{125}$ to its simplest form.

$$\sqrt[12]{125} = \sqrt[4]{\sqrt[3]{125}} = \sqrt[4]{5},\ \textit{Ans.}$$

2. Reduce $\sqrt[6]{8a^3}$ to its simplest form.

$$\sqrt[6]{8a^3} = \sqrt{\sqrt[3]{8a^3}} = \sqrt{2a},\ \textit{Ans.}$$

Hence the

RULE.

Resolve the index of the root into two factors, one of which is the same as the degree of the quantity under the radical sign. Extract this root of the quantity, and place the result under a radical whose index is the other factor.

EXAMPLES.

Reduce the following to their simplest form:

3. $\sqrt[4]{36}$.
8. $\sqrt[4]{324}$.
13. $\sqrt[4]{\frac{4}{9}}$.

4. $\sqrt[6]{49}$.
9. $\sqrt[6]{512}$.
14. $\sqrt[6]{\frac{27}{64}}$.

5. $\sqrt[6]{125}$.
10. $\sqrt[6]{256}$.
15. $\sqrt[mn]{x^{2n}y^{3n}}$.

6. $\sqrt[9]{8}$.
11. $\sqrt[9]{125}$.
16. $\sqrt[mn]{(a+b)^m}$.

7. $\sqrt[12]{81}$.
12. $\sqrt[9]{1728}$.
17. $\sqrt[2n]{a^2-2ab+b^2}$.

18. $\sqrt[4]{16a^2+96ay+144y^2}$.
20. $\sqrt[8]{(x^2-y^2)^4(x+y)^4}$.

19. $\sqrt[6]{x^8+2x^4y^2+y^4}$.
21. $\sqrt[9]{a^3-3a^2b+3ab^2-b^3}$.

194. To reduce a rational quantity to the form of a radical.

1. Reduce $3a^2x$ to the form of the square root.

$$\sqrt{(3a^2x)^2} = \sqrt{9a^4x^2}, \textit{ Ans.}$$

2. Introduce the coefficient of $3x\sqrt[3]{2a}$ under the radical sign.

$$3x\sqrt[3]{2a} = \sqrt[3]{(3x)^3 \times 2a} = \sqrt[3]{54x^3a}, \textit{ Ans.}$$

Hence the

RULE.

Raise the quantity to the required power, and place the result under the corresponding radical sign.

EXAMPLES.

Reduce the following to the form of the square root:

3. $2am$.
4. $4x^2y$.
5. $a + x$.
6. $a^2 - y$.
7. $3x - 2y$.
8. $A + B$.

Reduce the following to the form of the cube root:

9. $3ay$.
10. $4x^2y^3$.
11. $a - x$.
12. $a^{\frac{1}{2}}y^{\frac{1}{3}}$.
13. $(a - n)^4$.
14. $(x - y)^n$.

Express the following without a coefficient:

15. $3a\sqrt[2]{2a}$.
16. $\frac{4}{5}\sqrt{5ax}$.
17. $5a\sqrt[3]{mx}$.
18. $4x\sqrt{a - x}$.
19. $(x + y)\sqrt{2a}$.
20. $2x\sqrt{x^2 - y^2}$.
21. $(x - y)\sqrt{x - y}$.
22. $(x + y)\sqrt[3]{(x - y)^2}$.
23. $m^{\frac{1}{2}}n^2\sqrt{m - n}$.
24. $(a - b)\sqrt{\dfrac{x}{a - b}}$.
25. $(m - n)\sqrt{\dfrac{1}{m^2 - n^2}}$.
26. $(a + x)\sqrt[3]{\dfrac{1}{(a + x)^2}}$.
27. $\dfrac{m + n}{m - n}\sqrt{\dfrac{m - n}{m + n}}$.

195. To reduce radicals of different degrees to equivalent radicals of the same degree.

1. Reduce $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{6}$ to equivalent radicals of the same degree.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{2^6} = \sqrt[12]{64}.$$

$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{4}{12}} = \sqrt[12]{3^4} = \sqrt[12]{81}.$$

$$\sqrt[4]{6} = 6^{\frac{1}{4}} = 6^{\frac{3}{12}} = \sqrt[12]{6^3} = \sqrt[12]{216}.$$

Hence the

RULE.

Write the radicals with fractional exponents, reduce these exponents to a common denominator, and express the results in the radical form.

EXAMPLES.

Reduce the following to a common index:

2. $\sqrt{2}$ and $\sqrt[3]{5}$.
3. $\sqrt{a}$, $\sqrt[3]{b}$, and $\sqrt[4]{c}$.
4. $\sqrt[3]{2}$, $\sqrt[4]{3}$, and $\sqrt[5]{4}$.
5. $\sqrt[4]{an}$, $\sqrt[5]{bc}$, and $\sqrt{d}$.
6. $2\sqrt{2}$, $3\sqrt[3]{5}$, and $3\sqrt[3]{6}$.
7. $\sqrt[3]{a}$, $\sqrt[4]{b}$, and $\sqrt[6]{c}$.
8. $\sqrt[3]{a-b}$ and $\sqrt[4]{a+b}$.
9. $\sqrt[5]{a+x}$ and $\sqrt[6]{a-x}$.
10. $\sqrt[3]{a^2}$, $\sqrt[6]{b^4}$, and $2\sqrt[5]{c^3}$.
11. $\sqrt[n]{a}$, $\sqrt[m]{b}$, and $\sqrt[p]{c}$.
12. $\sqrt{a+x}$, $\sqrt[n]{a-x}$, and $\sqrt[m]{a^2-x^2}$.
13. $\sqrt{m+n}$, $\sqrt[3]{m-n}$, and $\sqrt[9]{m^2+n^2}$.
14. $\sqrt[2n]{a+b}$, $\sqrt[n]{a-x}$, and $\sqrt[3]{a^2-x^2}$.

ADDITION AND SUBTRACTION OF RADICALS.

196. To *add* or *subtract* radicals, they should be reduced to their simplest forms.

1. Find the sum of $\sqrt{27}$ and $\sqrt{75}$.

$$\begin{aligned} \sqrt{27} &= \sqrt{9 \times 3} = 3\sqrt{3} \text{ (Art. 188).} \\ \sqrt{75} &= \sqrt{25 \times 3} = 5\sqrt{3} \\ \hline & \qquad\qquad 8\sqrt{3}, \textit{ Ans.} \end{aligned}$$

2. Subtract $\sqrt{\frac{1}{2}}$ from $\sqrt{8}$.

$$\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2} \text{ (Art. 189)}$$
$$\sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{1}{4} \times 2} = \tfrac{1}{2}\sqrt{2}$$
$$1\tfrac{1}{2}\sqrt{2}, \textit{ Ans.}$$

3. Simplify $\sqrt{54} + \sqrt{\frac{3}{8}} - \sqrt[4]{576}$.

$$\begin{aligned}\sqrt{54} + \sqrt{\tfrac{3}{8}} - \sqrt[4]{576} &= \sqrt{9 \times 6} + \sqrt{\tfrac{1}{16} \times 6} - \sqrt{\sqrt{576}}\\ &= 3\sqrt{6} + \tfrac{1}{4}\sqrt{6} - 2\sqrt{6}\\ &= 1\tfrac{1}{4}\sqrt{6}, \textit{ Ans.}\end{aligned}$$

Hence, to add or subtract radicals, we have the following

RULE.

Reduce each radical to its simplest form, then unite the similar radicals, and indicate the addition or subtraction of the dissimilar radicals.

EXAMPLES.

Simplify the following:

4. $\sqrt{32} + \sqrt{50}$.
5. $\sqrt{48} + \sqrt{108}$.
6. $\sqrt{45} + \sqrt{80}$.
7. $\sqrt[3]{16} + \sqrt[3]{54}$.
8. $\sqrt{45} - \sqrt{20}$.
9. $\sqrt{243} - \sqrt{147}$.
10. $\sqrt[3]{875} - \sqrt[3]{448}$.
11. $\sqrt{\frac{3}{4}} - \sqrt{\frac{1}{3}}$.
12. $\frac{2}{5}\sqrt[3]{40} - 3\sqrt[3]{\frac{1}{25}}$.
13. $2\sqrt{\dfrac{3}{c}} - \sqrt{\dfrac{c}{3}}$.
14. $\sqrt{\dfrac{a+b}{a-b}} - \sqrt{\dfrac{a-b}{a+b}}$.
15. $\sqrt{45} + \sqrt{80} + \sqrt{125}$.
16. $\sqrt[3]{16} + \sqrt[3]{54} + \sqrt[3]{250}$.
17. $2\sqrt{32} + 3\sqrt{8} - 4\sqrt{2}$.
18. $3\sqrt{72} - 5\sqrt{8} - 2\sqrt{18}$.
19. $\sqrt{\frac{3}{2}a^3} + \sqrt{\frac{2}{3}a^3} - \sqrt{\frac{3}{8}a^3}$.
20. $\sqrt{1\frac{1}{3}} + 8\sqrt{12} - \frac{2}{3}\sqrt{108}$.
21. $5\sqrt[3]{54} - \sqrt[3]{128} + \sqrt[3]{250}$.
22. $\sqrt[6]{\frac{27}{64}} + \sqrt[4]{\frac{1}{9}} + \frac{1}{6}\sqrt[8]{81}$.
23. $\frac{1}{3}\sqrt{18x^2y^3} + \frac{1}{4}\sqrt{128x^2y^3}$.
24. $\sqrt{\dfrac{a}{4}} + \sqrt[4]{\dfrac{a^2}{81}} - \sqrt{\dfrac{a}{16}}$.
25. $\sqrt{\dfrac{x^3}{n^2}} - \sqrt{\dfrac{x^3}{m^2}} + \sqrt{\dfrac{x^3}{p^4}}$.

26. $\sqrt[3]{-81} - \sqrt[3]{-375} - \sqrt[3]{3}$.   28. $\sqrt[3]{\frac{1}{9}} + \sqrt[3]{\frac{1}{243}} + \sqrt[6]{\frac{4}{9}}$.

27. $\sqrt{\frac{3}{5}} + \sqrt{\frac{5}{3}} - \frac{1}{2}\sqrt[4]{225}$.   29. $\sqrt{12} + 5\sqrt{\frac{1}{27}} - \frac{2}{3}\sqrt[6]{27}$.

30. $2\sqrt{75} + 3\sqrt{147} + 4\sqrt{243}$.

31. $\frac{1}{2}\sqrt[3]{128} + \frac{2}{3}\sqrt[3]{432} + \frac{2}{5}\sqrt[3]{250}$.

32. $\sqrt{\frac{x+y}{x-y}} + \sqrt{\frac{x-y}{x+y}} - \frac{2y}{x-y}\sqrt{\frac{x-y}{x+y}}$.

33. $\frac{1}{\sqrt{\frac{1}{3}}} + \frac{1}{\sqrt{1\frac{1}{3}}} + \frac{1}{\sqrt{5\frac{1}{3}}}$.

34. $\sqrt{2a^3 + 4a^2b + 2ab^2} - \sqrt{2a^3 - 4a^2b + 2ab^2}$.

MULTIPLICATION OF RADICALS.

197. To multiply radicals, they must be of the same degree.

PRINCIPLE.

The product of the same roots of two quantities is equal to the same root of their product.

By Art. 189, $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

Whence, $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$.

1. Multiply $\sqrt{8}$ by $\sqrt{30}$.

$\sqrt{8} \times \sqrt{30} = \sqrt{8 \times 30}$ (Prin.) $= \sqrt{240} = 4\sqrt{15}$, *Ans.*

2. Multiply $\sqrt{a}$ by $\sqrt[3]{b}$.

$$\sqrt{a} = a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3},$$

$$\sqrt[3]{b} = b^{\frac{1}{3}} = b^{\frac{2}{6}} = \sqrt[6]{b^2}.$$

Then, $\sqrt{a} \times \sqrt[3]{b} = \sqrt[6]{a^3} \times \sqrt[6]{b^2} = \sqrt[6]{a^3b^2}$, *Ans.*

Hence, to multiply radicals, we have the following

RULE.

Reduce the radicals to a common index, if necessary.

Multiply the coefficients together for the coefficient of the product, and the quantities under the radical sign for the radical part of the product, and simplify the result.

EXAMPLES.

Multiply the following:

3. $\sqrt{3}$ by $\sqrt{5}$.
4. $\sqrt{a}$ by $\sqrt{x}$.
5. $\sqrt[3]{a^2x}$ by $\sqrt[3]{x^2}$.
6. $3\sqrt{6}$ by $5\sqrt{12}$.
7. $\sqrt[3]{a^2b}$ by $\sqrt[3]{ab^2}$.
8. $\sqrt{2}$ by $\sqrt[3]{3}$.
9. $\sqrt[4]{8}$ by $\sqrt{2}$.
10. $2\sqrt{a}$ by $3\sqrt[3]{a}$.
11. $\sqrt{a+b}$ by $\sqrt{a-b}$.
12. $3\sqrt[3]{108}$ by $4\sqrt[4]{128}$.
13. $2\sqrt{72}$ by $3\sqrt[5]{256}$.
14. $\sqrt{\frac{1}{2}}$ by $\sqrt[3]{\frac{1}{16}}$.
15. $\sqrt[4]{\frac{1}{8}}$ by $\sqrt[3]{\frac{1}{8}}$.
16. $\sqrt[4]{b}$ by $\sqrt[n]{a}$.
17. $\sqrt{2}$ by $\sqrt[n]{3}$.
18. $\sqrt[n]{a}$ by $\sqrt[m]{b}$.

Find the value of—

19. $\sqrt[3]{2ab^2} \times \sqrt[3]{3a^2b} \times \sqrt[3]{9ab}$.
20. $\sqrt[3]{ax} \times \sqrt{ax} \times \sqrt[3]{a^2x^2}$.
21. $\sqrt{2} \times \sqrt[3]{2} \times \sqrt[4]{2}$.
22. $\sqrt[3]{\frac{1}{2}} \times \sqrt{\frac{2}{3}} \times \sqrt{\frac{3}{4}}$.
23. $\sqrt{a-b} \times \sqrt{a+b} \times \sqrt{a^2-b^2}$.
24. $\sqrt{m+n} \times \sqrt{m-n} \times \sqrt[6]{m^2-n^2}$.
25. Multiply $2 + 4\sqrt{2}$ by $3 - 3\sqrt{2}$.

$$\begin{array}{l}
2 + 4\sqrt{2} \\
\underline{3 - 3\sqrt{2}} \\
6 + 12\sqrt{2} \\
\underline{- 6\sqrt{2} - 24} \\
6 + 6\sqrt{2} - 24 = 6\sqrt{2} - 18, \textit{ Ans.}
\end{array}$$

Multiply the following:

26. $3 + \sqrt{3}$ by $2 - 3\sqrt{3}$.
27. $2 - \sqrt{x}$ by $3 + \sqrt{x}$.
28. $3 + \sqrt{2}$ by $3 + \sqrt{2}$.
29. $\sqrt{2} - 3\sqrt{3}$ by $\sqrt{2} - 2\sqrt{3}$.

30. $2\sqrt{6} + \sqrt{5}$ by $2\sqrt{6} - \sqrt{5}$.

31. $\sqrt{5} + \sqrt{3} - \sqrt{2}$ by $\sqrt{5} + \sqrt{2}$.

32. $\sqrt{5} - \sqrt{3} - \sqrt{2}$ by $\sqrt{5} + \sqrt{3} - \sqrt{2}$.

33. $a + \sqrt{ab} + b$ by $a - \sqrt{ab} + b$.

34. $\sqrt{m} - \sqrt{n} - \sqrt{p}$ by $\sqrt{m} + \sqrt{n} + \sqrt{p}$.

DIVISION OF RADICALS.

198. To divide radicals, they must be of the same degree.

PRINCIPLE.

The quotient of the same roots of two quantities is equal to the same root of their quotient.

By ART. 189, $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

Whence, $\dfrac{\sqrt[n]{ab}}{\sqrt[n]{b}} = \sqrt[n]{a}.$

1. Divide $\sqrt{6}$ by $\sqrt{2}$.

$$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{\frac{6}{2}} \text{ (Prin.)} = \sqrt{3}, \textit{ Ans.}$$

2. Divide $\sqrt{ax}$ by $\sqrt[3]{a}$.

Reducing the radicals to a common index,

$$\sqrt{ax} = (ax)^{\frac{1}{2}} = (ax)^{\frac{3}{6}} = \sqrt[6]{a^3x^3}.$$

$$\sqrt[3]{a} = a^{\frac{1}{3}} = a^{\frac{2}{6}} = \sqrt[6]{a^2}.$$

Then, $\dfrac{\sqrt{ax}}{\sqrt[3]{a}} = \dfrac{\sqrt[6]{a^3x^3}}{\sqrt[6]{a^2}} = \sqrt[6]{\dfrac{a^3x^3}{a^2}} = \sqrt[6]{ax^3}$, *Ans.*

Hence, to divide radicals we have the following

RULE.

Reduce the radicals to a common index, if necessary.

Divide the coefficients for the coefficient of the quotient, and the quantities under the radical sign for the radical part of the quotient, *and simplify the result.*

EXAMPLES.

Divide the following:

3. $\sqrt{12}$ by $\sqrt{3}$.
4. $\sqrt{12}$ by $\sqrt{6}$.
5. $8\sqrt{45}$ by $2\sqrt{15}$.
6. $2\sqrt[3]{12}$ by $3\sqrt[3]{4}$.
7. $\sqrt{2}$ by $\sqrt{3}$.
8. $\sqrt{54}$ by $2\sqrt{9}$.
9. $\sqrt{1-a^2}$ by $\sqrt{1+a}$.
10. $18\sqrt{384}$ by $9\sqrt{3}$.
11. $\sqrt{12}$ by $\sqrt[3]{12}$.
12. $\sqrt{9}$ by $\sqrt[3]{9}$.
13. 8 by $\sqrt{8}$.
14. 5 by $\sqrt[3]{5}$.
15. $\sqrt[3]{16}$ by 4.
16. $\sqrt{3}$ by $\sqrt[4]{4}$.
17. $3\sqrt{7a}$ by $\sqrt[3]{7a}$.
18. $\sqrt[3]{a}$ by $\sqrt{a}$.
19. $\sqrt[4]{12}$ by $\sqrt{8}$.
20. $\sqrt[3]{xy}$ by $\sqrt{x^2y^2}$.
21. $\sqrt[5]{x}$ by $\sqrt[6]{x}$.
22. $a\sqrt[3]{x^2y^2}$ by $\sqrt[5]{x^2y}$.
23. $\sqrt{a^2-b^2}$ by $\sqrt[3]{a+b}$.
24. $\sqrt{15}-\sqrt{10}$ by $\sqrt{5}$.
25. $5-\sqrt{135}$ by $\sqrt{5}$.
26. $\sqrt{14}-\sqrt{35}$ by $\sqrt{2}-\sqrt{5}$.
27. $\sqrt{12}-\sqrt{27}$ by $\sqrt{2}-\sqrt{3}$.
28. $3-\sqrt[3]{3}+\sqrt[4]{3}$ by $\sqrt{3}$.
29. $a-b$ by $\sqrt[3]{a}-\sqrt[3]{b}$.
30. $a\sqrt{x}-\sqrt{bx}+a\sqrt{y}-\sqrt{by}$ by $\sqrt{x}+\sqrt{y}$.
31. $\sqrt{\dfrac{a-b}{a+b}}$ by $\sqrt{\dfrac{a+b}{a-b}}$.
32. $a+b-c+2\sqrt{ab}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

INVOLUTION OF RADICALS.

199. It is convenient in finding any power of a radical to use fractional exponents.

1. Raise $\sqrt[9]{x}$ to the third power.

$$(\sqrt[9]{x})^3 = (x^{\frac{1}{9}})^3 = x^{\frac{3}{9}} = x^{\frac{1}{3}} = \sqrt[3]{x}, \textit{ Ans.}$$

2. Raise $\sqrt[9]{x^2}$ to the fourth power.

$$(\sqrt[9]{x^2})^4 = (x^{\frac{2}{9}})^4 = x^{\frac{8}{9}} = \sqrt[9]{x^8}, \textit{ Ans.}$$

Hence, to raise a radical to any power, we have the following

RULE.

Either divide the index of the root by the exponent of the power, or raise the quantity under the radical sign to the required power.

Thus, $(\sqrt[6]{a})^3 = \sqrt{a}$.

$(\sqrt[5]{a})^4 = \sqrt[5]{a^4}$.

$(\sqrt[6]{18})^3 = \sqrt{18} = 3\sqrt{2}$.

EXAMPLES.

Find the values of the following:

3. $(\sqrt[4]{a})^2$.
4. $(3\sqrt[3]{ax})^3$
5. $(2\sqrt[12]{a^5x})^4$.
6. $(\sqrt[5]{a^2b})^6$.
7. $3\left(\sqrt{\frac{a}{3}}\right)^3$.
8. $(a\sqrt[4]{x^2})^n$.
9. $(x\sqrt[mn]{a-x})^m$.
10. $(\sqrt[12]{48})^4$.
11. $(2\sqrt[4]{12ab})^2$.
12. $(2+\sqrt{x})^2$.
13. $(\sqrt{3}+\sqrt{5})^2$.
14. $(\sqrt{a}+\sqrt{b})^2$.
15. $(a+\sqrt{x})^3$.
16. $(\sqrt[3]{2}-\sqrt[3]{3})^3$.
17. $(\sqrt{a}-\sqrt{b})^3$.
18. $(3+\sqrt{2})^2$.
19. $(\sqrt{5}-2)^2$.
20. $(\sqrt{6}-\sqrt{5})^2$
21. $(7-\sqrt{5})^2$.
22. $(\sqrt{7}+\sqrt{3})^2$.
23. $(\sqrt{8}+3)^2$.

EVOLUTION OF RADICALS.

200. It is convenient in finding any root of a radical to use fractional exponents.

1. Extract the cube root of $\sqrt[5]{a^6}$.

$$(\sqrt[5]{a^6})^{\frac{1}{3}} = (a^{\frac{6}{5}})^{\frac{1}{3}} = a^{\frac{2}{5}} = \sqrt[5]{a^2}, \textit{ Ans.}$$

2. Extract the cube root of $\sqrt[7]{a^2}$.

$$(\sqrt[7]{a^2})^{\frac{1}{3}} = (a^{\frac{2}{7}})^{\frac{1}{3}} = a^{\frac{2}{21}} = \sqrt[21]{a^2}, \textit{ Ans.}$$

Hence, to extract any root of a radical, we have the following

RULE.

Either extract the required root of the quantity under the radical sign, or multiply the index of the given root by the index of the required root.

$$\text{Thus, } (\sqrt[3]{a})^{\frac{1}{2}} = \sqrt{\sqrt[3]{a}} = \sqrt[6]{a}.$$

$$(\sqrt[5]{a^3})^{\frac{1}{3}} = \sqrt[5]{\sqrt[3]{a^3}} = \sqrt[5]{a}.$$

$$\left(\frac{a}{2}\sqrt[3]{\frac{a}{2}}\right)^{\frac{1}{4}} = \sqrt[3]{\sqrt[4]{\frac{a^4}{2^4}}} = \sqrt[3]{\frac{a}{2}} = \frac{1}{2}\sqrt[3]{4a}.$$

EXAMPLES.

Find the square root of—

3. $\sqrt[3]{a}$.
4. $\sqrt[3]{a^2b^2}$.
5. $\sqrt[5]{a^4b^2}$.
6. $25\sqrt[3]{2a}$.
7. $16\sqrt[5]{4a^4}$.
8. $64\sqrt[7]{a^6b^4}$.
9. $3\sqrt{7}$.
10. $100\sqrt[3]{36}$.
11. $a^2\sqrt[3]{81m^2n^2}$.

Find the value of—

12. $\sqrt{(\sqrt{2})}$.
13. $\sqrt[3]{(\sqrt{216})}$.
14. $\sqrt[5]{(\sqrt{243})}$.
15. $\sqrt[3]{(\sqrt[5]{64a^3n^3})}$.
16. $\sqrt[4]{\left(\frac{x}{2}\sqrt[3]{\frac{x}{2}}\right)}$.
17. $\sqrt{(\sqrt[3]{x+y})}$.
18. $\sqrt[5]{(2\sqrt{2})}$.
19. $\sqrt[4]{(a\sqrt[3]{a})}$.
20. $\sqrt[5]{(8\sqrt[3]{2})}$.
21. $\sqrt[3]{\left(\frac{a}{3}\sqrt{\frac{a}{3}}\right)}$.
22. $(a\sqrt[n]{\sqrt[3]{a^nb^n}})$.
23. $\sqrt[4]{(\sqrt[3]{ax})}$.
24. $\sqrt[3]{(\sqrt[5]{(a-b)^3y^3})}$.
25. $\sqrt[3]{(\sqrt{a^{3n}(a-b)^3})}$.
26. $\sqrt[6]{\left(\frac{a}{b}\sqrt[5]{\frac{a}{b}}\right)}$.
27. $\sqrt{(\sqrt[3]{a^2+2ab+b^2})}$.
28. $\sqrt[3]{(\sqrt{m^2(1-x)^6})}$.
29. $\sqrt[4]{(\sqrt[3]{a^8(x-y)^4})}$.
30. $\sqrt[5]{(\sqrt{a^5(1+x)^5})}$.
31. $\sqrt[n]{(\sqrt[3]{a^{2n}(a-b)^n})}$.
32. $\sqrt{(\sqrt[n]{a^4(1-x)^6})}$.

RATIONALIZATION.

201. Rationalization is the process of reducing a radical to a rational quantity.

202. To rationalize the denominator of a fraction.

CASE I.

When the denominator is a monomial.

1. Rationalize the denominator of $\frac{a}{\sqrt[3]{2x^2}}$.

Multiplying both terms by $\sqrt[3]{4x}$, we have

$$\frac{a}{\sqrt[3]{2x^2}} = \frac{a\sqrt[3]{4x}}{\sqrt[3]{2x^2} \times \sqrt[3]{4x}} = \frac{a\sqrt[3]{4x}}{\sqrt[3]{8x^3}} = \frac{a\sqrt[3]{4x}}{2x}, \text{ \textit{Ans.}}$$

From this example we derive the following

RULE.

Multiply both terms of the fraction by a factor that will render the denominator rational.

EXAMPLES.

Express the following with rational denominators:

2. $\frac{1}{\sqrt{2}}$.
3. $\frac{2}{\sqrt{a}}$.
4. $\frac{4}{3\sqrt{3}}$.
5. $\frac{a}{\sqrt[3]{b^2}}$.
6. $\frac{m}{a\sqrt{n}}$.
7. $\frac{5}{\sqrt[3]{9c^2}}$.
8. $\frac{2}{\sqrt[4]{8c^3}}$.
9. $\frac{4ay}{\sqrt[5]{16y^3}}$.
10. $\frac{3ax}{\sqrt[4]{27a^3}}$.
11. $\frac{m+n}{\sqrt[3]{36(m-n)^2}}$.
12. $\frac{\sqrt{1-c}}{\sqrt[3]{(1-c)^2}}$.
13. $\frac{\sqrt[3]{(a+b)}}{\sqrt[6]{(a+b)^5}}$.

CASE II.

When the denominator is a binomial surd of the second *degree.*

2. Subtract $\sqrt{\frac{1}{2}}$ from $\sqrt{8}$.

$$\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2} \text{ (Art. 189)}$$
$$\sqrt{\tfrac{1}{2}} = \sqrt{\tfrac{1}{4} \times 2} = \tfrac{1}{2}\sqrt{2}$$
$$1\tfrac{1}{2}\sqrt{2}, \textit{ Ans.}$$

3. Simplify $\sqrt{54} + \sqrt{\frac{3}{8}} - \sqrt[4]{576}$.

$$\sqrt{54} + \sqrt{\tfrac{3}{8}} - \sqrt[4]{576} = \sqrt{9 \times 6} + \sqrt{\tfrac{1}{16} \times 6} - \sqrt{\sqrt{576}}$$
$$= 3\sqrt{6} + \tfrac{1}{4}\sqrt{6} - 2\sqrt{6}$$
$$= 1\tfrac{1}{4}\sqrt{6}, \textit{ Ans.}$$

Hence, to add or subtract radicals, we have the following

RULE.

Reduce each radical to its simplest form, then unite the similar radicals, and indicate the addition or subtraction of the dissimilar radicals.

EXAMPLES.

Simplify the following:

4. $\sqrt{32} + \sqrt{50}$.

5. $\sqrt{48} + \sqrt{108}$.

6. $\sqrt{45} + \sqrt{80}$.

7. $\sqrt[3]{16} + \sqrt[3]{54}$.

8. $\sqrt{45} - \sqrt{20}$.

9. $\sqrt{243} - \sqrt{147}$.

10. $\sqrt[3]{875} - \sqrt[3]{448}$.

11. $\sqrt{\frac{3}{4}} - \sqrt{\frac{1}{8}}$.

12. $\frac{2}{5}\sqrt[3]{40} - 3\sqrt[3]{\frac{1}{25}}$.

13. $2\sqrt{\frac{3}{c}} - \sqrt{\frac{c}{3}}$.

14. $\sqrt{\frac{a+b}{a-b}} - \sqrt{\frac{a-b}{a+b}}$.

15. $\sqrt{45} + \sqrt{80} + \sqrt{125}$.

16. $\sqrt[3]{16} + \sqrt[3]{54} + \sqrt[3]{250}$.

17. $2\sqrt{32} + 3\sqrt{8} - 4\sqrt{2}$.

18. $3\sqrt{72} - 5\sqrt{8} - 2\sqrt{18}$.

19. $\sqrt{\frac{3}{2}a^3} + \sqrt{\frac{2}{3}a^3} - \sqrt{\frac{3}{8}a^3}$.

20. $\sqrt{1\frac{1}{3}} + 8\sqrt{12} - \frac{2}{3}\sqrt{108}$.

21. $5\sqrt[3]{54} - \sqrt[3]{128} + \sqrt[3]{250}$.

22. $\sqrt[6]{\frac{27}{64}} + \sqrt[4]{\frac{1}{9}} + \frac{1}{6}\sqrt[8]{81}$.

23. $\frac{1}{3}\sqrt{18x^2y^2} + \frac{1}{4}\sqrt{128x^2y^2}$.

24. $\sqrt{\frac{a}{4}} + \sqrt[4]{\frac{a^2}{81}} - \sqrt{\frac{a}{16}}$.

25. $\sqrt{\frac{x^3}{n^2}} - \sqrt{\frac{x^3}{m^2}} + \sqrt{\frac{x^3}{p^4}}$.

26. $\sqrt[3]{-81} - \sqrt[3]{-375} - \sqrt[3]{3}$.
28. $\sqrt[3]{\frac{1}{9}} + \sqrt[3]{\frac{1}{243}} + \sqrt[6]{\frac{4}{9}}$.

27. $\sqrt{\frac{3}{5}} + \sqrt{\frac{5}{3}} - \frac{1}{2}\sqrt[4]{225}$.
29. $\sqrt{12} + 5\sqrt{\frac{1}{27}} - \frac{2}{3}\sqrt[6]{27}$.

30. $2\sqrt{75} + 3\sqrt{147} + 4\sqrt{243}$.

31. $\frac{1}{2}\sqrt[3]{128} + \frac{2}{3}\sqrt[3]{432} + \frac{2}{5}\sqrt[3]{250}$.

32. $\sqrt{\frac{x+y}{x-y}} + \sqrt{\frac{x-y}{x+y}} - \frac{2y}{x-y}\sqrt{\frac{x-y}{x+y}}$.

33. $\frac{1}{\sqrt{\frac{1}{3}}} + \frac{1}{\sqrt{1\frac{1}{3}}} + \frac{1}{\sqrt{5\frac{1}{3}}}$.

34. $\sqrt{2a^3 + 4a^2b + 2ab^2} - \sqrt{2a^3 - 4a^2b + 2ab^2}$.

MULTIPLICATION OF RADICALS.

197. To multiply radicals, they must be of the same degree.

PRINCIPLE.

The product of the same roots of two quantities is equal to the same root of their product.

By Art. 189, $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.
Whence, $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$.

1. Multiply $\sqrt{8}$ by $\sqrt{30}$.

$\sqrt{8} \times \sqrt{30} = \sqrt{8 \times 30}$ (Prin.) $= \sqrt{240} = 4\sqrt{15}$, *Ans.*

2. Multiply $\sqrt{a}$ by $\sqrt[3]{b}$.

$$\sqrt{a} = a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3},$$
$$\sqrt[3]{b} = b^{\frac{1}{3}} = b^{\frac{2}{6}} = \sqrt[6]{b^2}.$$

Then, $\sqrt{a} \times \sqrt[3]{b} = \sqrt[6]{a^3} \times \sqrt[6]{b^2} = \sqrt[6]{a^3b^2}$, *Ans.*

Hence, to multiply radicals, we have the following

RULE.

Reduce the radicals to a common index, if necessary.

Multiply the coefficients together for the coefficient of the product, and the quantities under the radical sign for the radical part of the *product, and simplify the result.*

EXAMPLES.

Multiply the following:

3. $\sqrt{3}$ by $\sqrt{5}$.
4. $\sqrt{a}$ by $\sqrt{x}$.
5. $\sqrt[3]{a^2x}$ by $\sqrt[3]{x^2}$.
6. $3\sqrt{6}$ by $5\sqrt{12}$.
7. $\sqrt[3]{a^2b}$ by $\sqrt[3]{ab^2}$.
8. $\sqrt{2}$ by $\sqrt[3]{3}$.
9. $\sqrt[4]{8}$ by $\sqrt{2}$.
10. $2\sqrt{a}$ by $3\sqrt[3]{a}$.
11. $\sqrt{a+b}$ by $\sqrt{a-b}$.
12. $3\sqrt[3]{108}$ by $4\sqrt[4]{128}$.
13. $2\sqrt{72}$ by $3\sqrt[5]{256}$.
14. $\sqrt{\frac{1}{2}}$ by $\sqrt[3]{\frac{1}{16}}$.
15. $\sqrt[4]{\frac{1}{8}}$ by $\sqrt[3]{\frac{1}{8}}$.
16. $\sqrt[4]{b}$ by $\sqrt[n]{a}$.
17. $\sqrt{2}$ by $\sqrt[n]{3}$.
18. $\sqrt[n]{a}$ by $\sqrt[m]{b}$.

Find the value of—

19. $\sqrt[3]{2ab^2} \times \sqrt[3]{3a^2b} \times \sqrt[3]{9ab}$.
20. $\sqrt[3]{ax} \times \sqrt{ax} \times \sqrt[3]{a^2x^2}$.
21. $\sqrt{2} \times \sqrt[3]{2} \times \sqrt[4]{2}$.
22. $\sqrt[3]{\frac{1}{2}} \times \sqrt{\frac{2}{3}} \times \sqrt{\frac{3}{4}}$.
23. $\sqrt{a-b} \times \sqrt{a+b} \times \sqrt{a^2-b^2}$.
24. $\sqrt{m+n} \times \sqrt{m-n} \times \sqrt[6]{m^2-n^2}$.
25. Multiply $2 + 4\sqrt{2}$ by $3 - 3\sqrt{2}$.

$$
\begin{array}{l}
2 + 4\sqrt{2} \\
\underline{3 - 3\sqrt{2}} \\
6 + 12\sqrt{2} \\
\underline{ - 6\sqrt{2} - 24} \\
6 + 6\sqrt{2} - 24 = 6\sqrt{2} - 18, \textit{ Ans.}
\end{array}
$$

Multiply the following:

26. $3 + \sqrt{3}$ by $2 - 3\sqrt{3}$.
27. $2 - \sqrt{x}$ by $3 + \sqrt{x}$.
28. $3 + \sqrt{2}$ by $3 + \sqrt{2}$.
29. $\sqrt{2} - 3\sqrt{3}$ by $\sqrt{2} - 2\sqrt{3}$.

30. $2\sqrt{6} + \sqrt{5}$ by $2\sqrt{6} - \sqrt{5}$.

31. $\sqrt{5} + \sqrt{3} - \sqrt{2}$ by $\sqrt{5} + \sqrt{2}$.

32. $\sqrt{5} - \sqrt{3} - \sqrt{2}$ by $\sqrt{5} + \sqrt{3} - \sqrt{2}$.

33. $a + \sqrt{ab} + b$ by $a - \sqrt{ab} + b$.

34. $\sqrt{m} - \sqrt{n} - \sqrt{p}$ by $\sqrt{m} + \sqrt{n} + \sqrt{p}$.

DIVISION OF RADICALS.

198. To divide radicals, they must be of the same degree.

PRINCIPLE.

The quotient of the same roots of two quantities is equal to the same root of their quotient.

By Art. 189, $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$

Whence, $\dfrac{\sqrt[n]{ab}}{\sqrt[n]{b}} = \sqrt[n]{a}.$

1. Divide $\sqrt{6}$ by $\sqrt{2}$.

$$\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{\frac{6}{2}} \text{ (Prin.)} = \sqrt{3}, \textit{ Ans.}$$

2. Divide $\sqrt{ax}$ by $\sqrt[3]{a}$.

Reducing the radicals to a common index,

$$\sqrt{ax} = (ax)^{\frac{1}{2}} = (ax)^{\frac{3}{6}} = \sqrt[6]{a^3x^3}.$$

$$\sqrt[3]{a} = a^{\frac{1}{3}} = a^{\frac{2}{6}} = \sqrt[6]{a^2}.$$

Then, $\dfrac{\sqrt{ax}}{\sqrt[3]{a}} = \dfrac{\sqrt[6]{a^3x^3}}{\sqrt[6]{a^2}} = \sqrt[6]{\dfrac{a^3x^3}{a^2}} = \sqrt[6]{ax^3}$, *Ans.*

Hence, to divide radicals we have the following

RULE.

Reduce the radicals to a common index, if necessary.

Divide the coefficients for the coefficient of the quotient, and the quantities under the radical sign for the radical part of the quotient, and simplify the result.

EXAMPLES.

Divide the following:

3. $\sqrt{12}$ by $\sqrt{3}$.
4. $\sqrt{12}$ by $\sqrt{6}$.
5. $8\sqrt{45}$ by $2\sqrt{15}$.
6. $2\sqrt[3]{12}$ by $3\sqrt[3]{4}$.
7. $\sqrt{2}$ by $\sqrt{3}$.
8. $\sqrt{54}$ by $2\sqrt{9}$.
9. $\sqrt{1-a^2}$ by $\sqrt{1+a}$.
10. $18\sqrt{384}$ by $9\sqrt{3}$.
11. $\sqrt{12}$ by $\sqrt[3]{12}$.
12. $\sqrt{9}$ by $\sqrt[3]{9}$.
13. 8 by $\sqrt{8}$.
14. 5 by $\sqrt[3]{5}$.
15. $\sqrt[3]{16}$ by 4.
16. $\sqrt{3}$ by $\sqrt[4]{4}$.
17. $3\sqrt{7a}$ by $\sqrt[3]{7a}$.
18. $\sqrt[3]{a}$ by $\sqrt{a}$.
19. $\sqrt[4]{12}$ by $\sqrt{8}$.
20. $\sqrt[3]{xy}$ by $\sqrt{x^2y^2}$.
21. $\sqrt[5]{x}$ by $\sqrt[6]{x}$.
22. $a\sqrt[3]{x^2y^2}$ by $\sqrt[5]{x^2y}$.
23. $\sqrt{a^2-b^2}$ by $\sqrt[3]{a+b}$.
24. $\sqrt{15}-\sqrt{10}$ by $\sqrt{5}$.
25. $5-\sqrt{135}$ by $\sqrt{5}$.
26. $\sqrt{14}-\sqrt{35}$ by $\sqrt{2}-\sqrt{5}$.
27. $\sqrt{12}-\sqrt{27}$ by $\sqrt{2}-\sqrt{3}$.
28. $3-\sqrt[3]{3}+\sqrt[4]{3}$ by $\sqrt{3}$.
29. $a-b$ by $\sqrt[3]{a}-\sqrt[3]{b}$.
30. $a\sqrt{x}-\sqrt{bx}+a\sqrt{y}-\sqrt{by}$ by $\sqrt{x}+\sqrt{y}$.
31. $\sqrt{\dfrac{a-b}{a+b}}$ by $\sqrt{\dfrac{a+b}{a-b}}$.
32. $a+b-c+2\sqrt{ab}$ by $\sqrt{a}+\sqrt{b}-\sqrt{c}$.

INVOLUTION OF RADICALS.

199. It is convenient in finding any power of a radical to use fractional exponents.

1. Raise $\sqrt[9]{x}$ to the third power.

$$(\sqrt[9]{x})^3 = (x^{\frac{1}{9}})^3 = x^{\frac{3}{9}} = x^{\frac{1}{3}} = \sqrt[3]{x}, \textit{ Ans.}$$

2. Raise $\sqrt[9]{x^2}$ to the fourth power.

$$(\sqrt[9]{x^2})^4 = (x^{\frac{2}{9}})^4 = x^{\frac{8}{9}} = \sqrt[9]{x^8}, \textit{ Ans.}$$

Hence, to raise a radical to any power, we have the following

RULE.

Either divide the index of the root by the exponent of the power, or raise the quantity under the radical sign to the required power.

Thus, $(\sqrt[6]{a})^3 = \sqrt{a}$.

$$(\sqrt[5]{a})^4 = \sqrt[5]{a^4}.$$

$$(\sqrt[6]{18})^3 = \sqrt{18} = 3\sqrt{2}.$$

EXAMPLES.

Find the values of the following:

3. $(\sqrt[4]{a})^2$.
4. $(3\sqrt[3]{ax})^3$
5. $(2\sqrt[12]{a^5x})^4$.
6. $(\sqrt[5]{a^2b})^6$.
7. $3\left(\sqrt{\frac{a}{3}}\right)^3$.
8. $(a\sqrt[4]{x^2})^n$.
9. $(x\sqrt[mn]{a-x})^m$.
10. $(\sqrt[12]{48})^4$.
11. $(2\sqrt[4]{12ab})^2$.
12. $(2+\sqrt{x})^2$.
13. $(\sqrt{3}+\sqrt{5})^2$.
14. $(\sqrt{a}+\sqrt{b})^2$.
15. $(a+\sqrt{x})^3$.
16. $(\sqrt[3]{2}-\sqrt[3]{3})^3$.
17. $(\sqrt{a}-\sqrt{b})^3$.
18. $(3+\sqrt{2})^2$.
19. $(\sqrt{5}-2)^2$.
20. $(\sqrt{6}-\sqrt{5})^2$
21. $(7-\sqrt{5})^2$.
22. $(\sqrt{7}+\sqrt{3})^2$.
23. $(\sqrt{8}+3)^2$.

EVOLUTION OF RADICALS.

200. It is convenient in finding any root of a radical to use fractional exponents.

1. Extract the cube root of $\sqrt[5]{a^6}$.

$$(\sqrt[5]{a^6})^{\frac{1}{3}} = (a^{\frac{6}{5}})^{\frac{1}{3}} = a^{\frac{2}{5}} = \sqrt[5]{a^2}, \textit{ Ans.}$$

2. Extract the cube root of $\sqrt[7]{a^2}$.

$$(\sqrt[7]{a^2})^{\frac{1}{3}} = (a^{\frac{2}{7}})^{\frac{1}{3}} = a^{\frac{2}{21}} = \sqrt[21]{a^2}, \textit{ Ans.}$$

Hence, to extract any root of a radical, we have the following

RULE.

Either extract the required root of the quantity under the radical sign, or multiply the index of the given root by the index of the required root.

$$\text{Thus, } (\sqrt[3]{a})^{\frac{1}{2}} = \sqrt{\sqrt[3]{a}} = \sqrt[6]{a}.$$

$$(\sqrt[5]{a^3})^{\frac{1}{3}} = \sqrt[5]{\sqrt[3]{a^3}} = \sqrt[5]{a}.$$

$$\left(\frac{a}{2}\sqrt[3]{\frac{a}{2}}\right)^{\frac{1}{4}} = \sqrt[3]{\sqrt[4]{\frac{a^4}{2^4}}} = \sqrt[3]{\frac{a}{2}} = \frac{1}{2}\sqrt[3]{4a}.$$

EXAMPLES.

Find the square root of—

3. $\sqrt[3]{a}$.
4. $\sqrt[3]{a^2b^2}$.
5. $\sqrt[5]{a^4b^2}$.
6. $25\sqrt[3]{2a}$.
7. $16\sqrt[5]{4a^4}$.
8. $64\sqrt[7]{a^6b^4}$.
9. $3\sqrt{7}$.
10. $100\sqrt[3]{36}$.
11. $a^2\sqrt[3]{81m^2n^2}$.

Find the value of—

12. $\sqrt{(\sqrt{2})}$.
13. $\sqrt[3]{(\sqrt{216})}$.
14. $\sqrt[5]{(\sqrt{243})}$.
15. $\sqrt[3]{(\sqrt[5]{64a^3n^3})}$.
16. $\sqrt[4]{\left(\frac{x}{2}\sqrt[3]{\frac{x}{2}}\right)}$.
17. $\sqrt{(\sqrt[3]{x+y})}$.
18. $\sqrt[5]{(2\sqrt{2})}$.
19. $\sqrt[4]{(a\sqrt[3]{a})}$.
20. $\sqrt[5]{(8\sqrt[3]{2})}$.
21. $\sqrt[3]{\left(\frac{a}{3}\sqrt{\frac{a}{3}}\right)}$.
22. $(a\sqrt[n]{\sqrt[3]{a^nb^n}})$.
23. $\sqrt[4]{(\sqrt[3]{ax})}$.
24. $\sqrt[3]{(\sqrt[5]{(a-b)^3y^3})}$.
25. $\sqrt[3]{(\sqrt{a^{3n}(a-b)^3})}$.
26. $\sqrt[6]{\left(\frac{a}{b}\sqrt[5]{\frac{a}{b}}\right)}$.
27. $\sqrt{(\sqrt[3]{a^2+2ab+b^2})}$.
28. $\sqrt[3]{(\sqrt{m^2(1-x)^6})}$.
29. $\sqrt[4]{(\sqrt[3]{a^8(x-y)^4})}$.
30. $\sqrt[5]{(\sqrt{a^5(1+x)^5})}$.
31. $\sqrt[n]{(\sqrt[3]{a^{2n}(a-b)^n})}$.
32. $\sqrt{(\sqrt[n]{a^4(1-x)^6})}$.

RATIONALIZATION.

201. Rationalization is the process of reducing a radical to a rational quantity.

202. To rationalize the denominator of a fraction.

CASE I.

When the denominator is a monomial.

1. Rationalize the denominator of $\frac{a}{\sqrt[3]{2x^2}}$.

Multiplying both terms by $\sqrt[3]{4x}$, we have

$$\frac{a}{\sqrt[3]{2x^2}} = \frac{a\sqrt[3]{4x}}{\sqrt[3]{2x^2} \times \sqrt[3]{4x}} = \frac{a\sqrt[3]{4x}}{\sqrt[3]{8x^3}} = \frac{a\sqrt[3]{4x}}{2x}, \text{ Ans.}$$

From this example we derive the following

RULE.

Multiply both terms of the fraction by a factor that will render the denominator rational.

EXAMPLES.

Express the following with rational denominators:

2. $\frac{1}{\sqrt{2}}$.

3. $\frac{2}{\sqrt{a}}$.

4. $\frac{4}{3\sqrt{3}}$.

5. $\frac{a}{\sqrt[3]{b^2}}$.

6. $\frac{m}{a\sqrt{n}}$.

7. $\frac{5}{\sqrt[3]{9c^2}}$.

8. $\frac{2}{\sqrt[4]{8c^3}}$.

9. $\frac{4ay}{\sqrt[5]{16y^3}}$.

10. $\frac{3ax}{\sqrt[4]{27a^3}}$.

11. $\frac{m+n}{\sqrt[3]{36(m-n)^2}}$.

12. $\frac{\sqrt{1-c}}{\sqrt[3]{(1-c)^2}}$.

13. $\frac{\sqrt[3]{(a+b)}}{\sqrt[6]{(a+b)^5}}$.

CASE II.

When the denominator is a binomial surd of the second degree.

1. Rationalize the denominator of $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

Multiplying both terms by $\sqrt{a}-\sqrt{b}$, we have

$$\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{(\sqrt{a}-\sqrt{b})^2}{(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})}=\frac{(\sqrt{a}-\sqrt{b})^2}{a-b},\ \textit{Ans.}$$

From this example we derive the following

RULE.

Multiply both terms of the fraction by the denominator, with the sign between the terms changed.

EXAMPLES.

Express the following with rational denominators.

2. $\dfrac{4}{3+\sqrt{2}}$.

3. $\dfrac{6}{\sqrt{3}-1}$.

4. $\dfrac{3-\sqrt{2}}{2-\sqrt{3}}$.

5. $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

6. $\dfrac{\sqrt{7}+2}{\sqrt{7}-2}$.

7. $\dfrac{\sqrt{5}-1}{\sqrt{5}+1}$.

8. $\dfrac{3\sqrt{3}+2\sqrt{2}}{3\sqrt{3}-2\sqrt{2}}$.

9. $\dfrac{\sqrt{m}+\sqrt{n}}{\sqrt{m}-\sqrt{n}}$.

10. $\dfrac{x-\sqrt{y}}{x+\sqrt{y}}$.

11. $\dfrac{\sqrt{m-n}-\sqrt{n}}{\sqrt{m-n}+\sqrt{n}}$.

12. $\dfrac{\sqrt{2}-\sqrt{1-x}}{\sqrt{2}+\sqrt{1-x}}$.

13. $\dfrac{a+\sqrt{a^2+2}}{a-\sqrt{a^2+2}}$.

14. $\dfrac{\sqrt{a+2}-\sqrt{a}}{\sqrt{a+2}+\sqrt{a}}$.

15. $\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$.

16. $\dfrac{\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}{\sqrt{x^2+x+1}-\sqrt{x^2-x+1}}$.

Note.—Some binomial surds of higher degree than the second can be rationalized. Thus, the denominator of $\dfrac{a}{\sqrt[3]{a}-\sqrt[3]{b}}$ can be rationalized by multiplying both terms by $\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}$.

$$\frac{a}{\sqrt[3]{a}-\sqrt[3]{b}}=\frac{a(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})}{(\sqrt[3]{a}-\sqrt[3]{b})(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})}=\frac{a(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})}{a-b}.$$

203. To find what factor will rationalize a binomial surd, divide the surd into any binomial that is rational, which will exactly contain the surd.

Thus, if the denominator is $\sqrt[3]{m}+\sqrt[3]{n}$, divide $m+n$ by $m^{\frac{1}{3}}+n^{\frac{1}{3}}$. (See ART. 182, Prob. 16.)

Express the following with rational denominators:

1. $\dfrac{4}{\sqrt[3]{3}+\sqrt[3]{2}}$.
2. $\dfrac{\sqrt[3]{m}}{\sqrt[3]{m}-n}$.
3. $\dfrac{2-\sqrt[3]{3}}{2+\sqrt[3]{3}}$.
4. $\dfrac{\sqrt[3]{4}+\sqrt[3]{3}}{\sqrt[3]{4}-\sqrt[3]{3}}$.
5. $\dfrac{3}{\sqrt[4]{a}-\sqrt[4]{b}}$.
6. $\dfrac{\sqrt[3]{a^2}-\sqrt[3]{b^2}}{\sqrt[3]{a^2}+\sqrt[3]{b^2}}$.

IMAGINARY QUANTITIES.

204. An **Imaginary Quantity** is an indicated even root of a negative quantity; as, $\sqrt{-2}$, or $\sqrt[4]{-a}$.

All other quantities are called *real* quantities.

PRINCIPLE.

Every imaginary quantity may be reduced to the form of $a\sqrt{-1}$ or $a\sqrt[2n]{-1}$.

Thus, $\sqrt{-x^2}=\sqrt{x^2\times-1}=\sqrt{x^2}\times\sqrt{-1}=\pm x\sqrt{-1}$;

$\sqrt{-3}=\sqrt{3\times-1}=\sqrt{3}\times\sqrt{-1}$;

or, putting a for the coefficient, we have $\pm a\sqrt{-1}$.

In all higher powers the form will be $\pm a\sqrt[2n]{-1}$.

NOTE.—This is evident, since if n equals 1, 2, 3, 4, etc., $2n$ will equal 2, 4, 6, 8, etc.; hence, $2n$ represents all even numbers.

205. To add or subtract imaginary quantities.

1. Add $\sqrt{-4}$ and $\sqrt{-16}$.

$$\begin{array}{rl} \text{By ART. 190, } \sqrt{-4} &= 2\sqrt{-1} \\ \sqrt{-16} &= 4\sqrt{-1} \\ \hline & 6\sqrt{-1}, \textit{ Ans.} \end{array}$$

Find the values of—

2. $\sqrt{-4}+\sqrt{-9}$.
3. $\sqrt{-36}+\sqrt{-100}$.
4. $\sqrt{-121}-\sqrt{-81}$.
5. $\sqrt{-a^2}-\sqrt{-b^2}$.
6. $\sqrt{-49m^2}-\sqrt{-25m^2}$.
7. $\sqrt{-300}-\sqrt{-108}$.
8. $\sqrt{-32a^2}+a\sqrt{-8}$.
9. $\sqrt{-18b^2}-c\sqrt{-50}$.
10. $\sqrt{-(a+b)^2}+\sqrt{-(a-b)^2}$.
11. $\sqrt{-m^2+2mn-n^2}+\sqrt{-(m+n)^2}$.

206. To multiply imaginary quantities.

PRINCIPLE.

The **product** *of two imaginary quantities is real, with the sign before the result the* **reverse** *of that obtained by the common rule for multiplication.*

Thus, $\sqrt{-a^2}\times\sqrt{-b^2}=a\sqrt{-1}\times b\sqrt{-1}=ab\times-1=-ab$

$$-\sqrt{-a^2}\times\sqrt{-b^2}=-a\sqrt{-1}\times b\sqrt{-1}=-ab\times-1=ab.$$

1. Multiply $2\sqrt{-2}$ by $3\sqrt{-3}$.

By Art. 190,

$$\begin{array}{l} 2\sqrt{-2}=2\sqrt{2}\times\sqrt{-1} \\ \underline{3\sqrt{-3}=3\sqrt{3}\times\sqrt{-1}} \\ \qquad 6\sqrt{6}\times-1=-6\sqrt{6}, \textit{Ans.} \end{array}$$

Multiply—

2. $\sqrt{-2}$ by $\sqrt{-3}$.
3. $\sqrt{-3}$ by $-\sqrt{-15}$.
4. $-\sqrt{-18}$ by $-\sqrt{-2}$.
5. $-\sqrt{-5}$ by $\sqrt{-3}$.
6. $2\sqrt{-3}$ by $3\sqrt{-12}$.
7. $1+\sqrt{-1}$ by $1+\sqrt{-1}$.
8. $1-\sqrt{-1}$ by $1-\sqrt{-1}$.
9. $1+\sqrt{-1}$ by $1-\sqrt{-1}$.
10. $\sqrt{a}+\sqrt{-b}$ by $\sqrt{a}-\sqrt{-b}$.
11. $a+b\sqrt{-1}$ by $a-b\sqrt{-1}$.
12. $\sqrt{-2}$ by $\sqrt{-3}\times\sqrt{-4}$.
13. $(\sqrt{a}+\sqrt{-b})(\sqrt{-a}+\sqrt{b})$ by $-\sqrt{-a^2}$.
14. $2+3\sqrt{-3}$ by $3-2\sqrt{-3}$.

Expand the following:

15. $(1-\sqrt{-2})^2$.
16. $(\sqrt{-3}-\sqrt{-2})^2$.
17. $(\sqrt{-m}-\sqrt{-n})^2$.
18. $(\sqrt{-3}+\sqrt{2})^2$.
19. $(a-\sqrt{-b})^3$.
20. $(\sqrt{-a}+b)^3$.
21. $(\sqrt{-a}-\sqrt{-b})^3$.
22. $(\sqrt{2}+\sqrt{-3})^3$.

207. To divide imaginary quantities.

PRINCIPLE.

The **quotient** *of two imaginary quantities is real, with the sign before the result the* **same** *as that obtained by the common rule for division.*

Thus,
$$\frac{\sqrt{-ab}}{\sqrt{-b}}=\frac{\sqrt{ab}\times\sqrt{-1}}{\sqrt{b}\times\sqrt{-1}}=\sqrt{a}.$$

$$\frac{-\sqrt{-ab}}{\sqrt{-b}}=\frac{-\sqrt{ab}\times\sqrt{-1}}{\sqrt{b}\times\sqrt{-1}}=-\sqrt{a}.$$

1. Divide $10\sqrt{-6}$ by $5\sqrt{-2}$.

$$\frac{10\sqrt{-6}}{5\sqrt{-2}}=\frac{10\sqrt{6}\times\sqrt{-1}}{5\sqrt{2}\times\sqrt{-1}}=2\sqrt{3}.$$

2. Divide $2\sqrt{-1}$ by $1+\sqrt{-1}$.

$$2\sqrt{-1}\div 1+\sqrt{-1}=\frac{2\sqrt{-1}}{1+\sqrt{-1}}.$$

Rationalizing the denominator,

$$\frac{2\sqrt{-1}\,(1-\sqrt{-1})}{(1+\sqrt{-1})\,(1-\sqrt{-1})}=\frac{2\sqrt{-1}+2}{2}=1+\sqrt{-1}.$$

Divide—

3. $3\sqrt{-10}$ by $\sqrt{-2}$.
4. $8\sqrt{-12}$ by $\sqrt{-3}$.
5. $a\sqrt{-b^2}$ by $-\sqrt{-b}$.
6. $\sqrt[4]{-8}$ by $\sqrt[4]{-2}$.
7. $\sqrt[6]{-36}$ by $\sqrt[6]{-4}$.
8. $\sqrt[8]{-48}$ by $\sqrt[8]{-3}$.
9. 2 by $1-\sqrt{-1}$.
10. $-2\sqrt{-1}$ by $1-\sqrt{-1}$.

11. $\sqrt{15}$ by $\sqrt{-3}$.

12. $\sqrt{18}$ by $\sqrt{-2}$.

13. $a + b$ by $\sqrt{a} - \sqrt{-b}$.

14. 1 by $\sqrt{-2} + \sqrt{-3}$.

15. $24 + 5\sqrt{-3}$ by $2 + 3\sqrt{-3}$.

16. $9 + \sqrt{15}$ by $\sqrt{-3} + \sqrt{-5}$.

PROPERTIES OF QUADRATIC SURDS.

208. A **Quadratic Surd** is the indicated square root of an imperfect square, as $\sqrt{a}$, $\sqrt{5}$, or $\sqrt{a+b}$.

PRINCIPLE.

A quadratic surd cannot equal the sum of a rational quantity and a quadratic surd.

If possible, let $\sqrt{a} = b + \sqrt{c}$

Squaring, $a = b^2 + 2b\sqrt{c} + c$

Or, $2b\sqrt{c} = a - b^2 - c$

And, $\sqrt{c} = \dfrac{a - b^2 - c}{2b}$

That is, a surd is equal to a rational quantity, which is impossible. Hence $\sqrt{a}$ cannot equal $b + \sqrt{c}$.

209. *If $a + \sqrt{b} = c + \sqrt{d}$, then $a = c$ and $\sqrt{b} = \sqrt{d}$.*

If a does not equal c, let $a = c \pm x$,

Substituting, $c \pm x + \sqrt{b} = c + \sqrt{d}$

Or, $\pm x + \sqrt{b} = \sqrt{d}$,

which is impossible by ART. 208, Prin.

Hence, $a = c$, and $\sqrt{b} = \sqrt{d}$.

210. *If $\sqrt{x} + \sqrt{y} = \sqrt{a + \sqrt{b}}$, then $\sqrt{x} - \sqrt{y} = \sqrt{a - \sqrt{b}}$.*

Squaring $\sqrt{x} + \sqrt{y} = \sqrt{a + \sqrt{b}}$, we have,

$x + 2\sqrt{xy} + y = a + \sqrt{b}$,

By ART. 209, $x + y = a$,

And, $2\sqrt{xy} = \sqrt{b}$

Subtracting, $x - 2\sqrt{xy} + y = a - \sqrt{b}$

Extracting the square root, $\sqrt{x} - \sqrt{y} = \sqrt{a - \sqrt{b}}$.

SQUARE ROOT OF A BINOMIAL SURD.

211. The preceding principles furnish an easy method of extracting the square root of binomial surds of the form

$$a + \sqrt{b}.$$

1. Extract the square root of $7 + 4\sqrt{3}$.

$$\begin{array}{lrcll}
\text{Let} & \sqrt{x} + \sqrt{y} & = & \sqrt{7 + 4\sqrt{3}} & (1) \\
\text{Then, by Art. 210,} & \sqrt{x} - \sqrt{y} & = & \sqrt{7 - 4\sqrt{3}} & (2) \\
\hline
\text{Multiplying,} & x - y & = & \sqrt{49 - 48} & (3) \\
\text{Or,} & x - y & = & 1 & (4) \\
\text{Squaring (1),} & x + 2\sqrt{xy} + y & = & 7 + 4\sqrt{3} & (5) \\
\text{By Art. 209,} & x + y & = & 7 & (6) \\
\text{Adding (4) and (6),} & 2x & = & 8 & \\
 & x & = & 4 & \\
\text{Subtracting (4) from (6),} & 2y & = & 6 & \\
 & y & = & 3 & \\
\text{Substituting in (1),} & \sqrt{7 + 4\sqrt{3}} & = & 2 + \sqrt{3},\ \textit{Ans.} &
\end{array}$$

212. Examples like the above may be solved by inspection, by expressing them in the form

$$x + 2\sqrt{xy} + y,$$

in which the coefficient of the surd term is 2, and the rational term is separated into two parts whose product is equal to the quantity under the radical sign. Then extract the square root of these parts, and connect them by the sign of the surd.

1. Extract the square root of $14 + 6\sqrt{5}$.

$$\sqrt{14 + 6\sqrt{5}} = \sqrt{14 + 2\sqrt{45}}$$

We separate 14 into two parts whose product is 45. These parts are 9 and 5; hence,

$$\begin{aligned}
\sqrt{14 + 2\sqrt{45}} &= \sqrt{9 + 2\sqrt{45} + 5} \\
&= 3 + \sqrt{5},\ \textit{Ans.}
\end{aligned}$$

2. Extract the square root of $28 + 10\sqrt{3}$.

$$\sqrt{28 + 10\sqrt{3}} = \sqrt{28 + 2\sqrt{75}}$$

We separate 28 into two parts whose product is 75. These parts are 25 and 3; hence,

$$\sqrt{28 + 2\sqrt{75}} = \sqrt{25 + 2\sqrt{75} + 3}$$
$$= 5 + \sqrt{3}, \textit{ Ans.}$$

EXAMPLES.

Extract the square root of—

3. $11 + 4\sqrt{7}$.
4. $10 + 4\sqrt{6}$.
5. $14 + 4\sqrt{10}$.
6. $11 - 6\sqrt{2}$.
7. $29 + 12\sqrt{5}$.
8. $14 - \sqrt{180}$.
9. $5 + \sqrt{24}$.
10. $30 - 12\sqrt{6}$.
11. $1 + 4\sqrt{-3}$.
12. $16 + \sqrt{252}$.
13. $28 - 4\sqrt{48}$.
14. $43 - \sqrt{1800}$.
15. $120 - \sqrt{8000}$.
16. $12 - \sqrt{108}$.
17. $47 - 6\sqrt{10}$.
18. $28 - \sqrt{300}$.
19. $7 - 6\sqrt{-2}$.
20. $16 + 4\sqrt{7}$.
21. $19 - 6\sqrt{10}$.
22. $+\sqrt{-1}$.
23. $\sqrt{18} - 4$.
24. $3\sqrt{6} + 2\sqrt{12}$.
25. $4\sqrt{6} + 6\sqrt{2}$.
26. $3\sqrt{5} + 2\sqrt{10}$.
27. $3\sqrt{3} - 2\sqrt{6}$.
28. $6\sqrt{-1}$.
29. $-4\sqrt{-1}$.
30. $a^2 - 2x\sqrt{a^2 - x^2}$.
31. $b + 2a\sqrt{b - a^2}$.
32. $2m - 2\sqrt{m^2 - x^2}$.
33. $6 - m - 4\sqrt{2 - m}$.

RADICAL EQUATIONS.

213. Radical Equations are those which contain the unknown quantity in the form of a radical.

A radical equation is solved by rationalizing the terms containing the unknown quantity, and then determining its value.

1. Solve the equation $\sqrt{x} + 5 = 14$.

Transposing, $\sqrt{x} = 9$

Squaring, $x = 81$, *Ans.*

2. Solve the equation $\sqrt{2x+5}+\sqrt{2x-3}=4$.

Transposing, $\sqrt{2x+5} = 4 - \sqrt{2x-3}$

Squaring, $2x + 5 = 16 - 8\sqrt{2x-3} + 2x - 3$

Transposing and uniting, $8\sqrt{2x-3} = -8$

Or, $\sqrt{2x-3} = -1$

Squaring, $2x - 3 = 1$

$2x = 4$

$x = 2$, *Ans.*

3. Solve the equation $\dfrac{\sqrt{2x+2}+\sqrt{2x-2}}{\sqrt{2x+2}-\sqrt{2x-2}}=4$.

Rationalizing the denominator, we have

$$\frac{4x + 2\sqrt{4x^2 - 4}}{4} = 4$$

Or, $4x + 2\sqrt{4x^2-4} = 16$

Transposing $4x$, and dividing by 2,

$$\sqrt{4x^2 - 4} = 8 - 2x$$

Squaring, $4x^2 - 4 = 64 - 32x + 4x^2$

Collecting, $32x = 68$

$x = 2\frac{1}{8}$, *Ans.*

EXAMPLES.

Solve the following equations:

4. $\sqrt{3x} + 2 = 8$.

5. $\sqrt{3x+4} + 1 = 5$.

6. $\sqrt{2x+2} - 5 = 1$.

7. $6 = \sqrt[3]{4x} + 4$.

8. $\sqrt{2x+5} = \sqrt{2x} + 1$.

9. $\sqrt{4x^2-3} = 2x - 3$.

10. $\sqrt{9x^2-20} = 3x - 2$.

11. $5x - 2 = \sqrt{25x^2-36}$.

12. $\sqrt{x+a} = \dfrac{3a}{\sqrt{x+a}}$.

13. $2\sqrt{x} + 3 = \sqrt{4x+11}$.

14. $\sqrt{x^2-5x+10} = x - 1$.

15. $\sqrt{x-4} + \sqrt{x+9} = 13$.

16. $\sqrt{x-11} + \sqrt{x+4} = 15$.

17. $\sqrt{x-20} = 10 - \sqrt{x}$.

18. $\sqrt{x-8} = 6 - \sqrt{x+4}$.

19. $\sqrt{x^2+4\sqrt{x^2+13}} = x + 2$.

20. $\sqrt{m^2+x} + \sqrt{m^2-x} = 2n$.

21. $\dfrac{x+a}{2\sqrt{x}} = \dfrac{\sqrt{4x}}{3}$.

22. $\sqrt{x+5}+\sqrt{x-5}=\sqrt{2x+4}$.

23. $\dfrac{\sqrt{x}+5}{\sqrt{x}+6}=\dfrac{\sqrt{x}-7}{\sqrt{x}-8}$.

24. $x+\sqrt{2-x}=\dfrac{2}{\sqrt{2-x}}$.

25. $\dfrac{\sqrt{x}+25}{\sqrt{x}+7}=\dfrac{\sqrt{x}+4}{\sqrt{x}+1}$.

26. $\sqrt{2x}+\sqrt{2x+9}=\dfrac{45}{\sqrt{2x+9}}$.

27. $\sqrt{x}-\sqrt{x-8}=\dfrac{6}{\sqrt{x}}$.

28. $\dfrac{1}{\sqrt{x-1}-1}+\dfrac{1}{\sqrt{x-1}+1}=\dfrac{3}{\sqrt{x-1}}$.

29. $\dfrac{x-4}{2+\sqrt{x}}=2\sqrt{x}-9+\dfrac{4\sqrt{x}}{3}$.

30. $\dfrac{n+x}{\sqrt{n+x}+\sqrt{n}}+\dfrac{n+x}{\sqrt{n+x}-\sqrt{n}}=4\sqrt{n+x}$.

31. $x-6=\dfrac{x^2}{(1+\sqrt{1+x})^2}$.

32. $\sqrt{2-\sqrt{4-x^2}}=\dfrac{\sqrt{5}-1}{2}$.

33. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=m$.

34. $\sqrt{\dfrac{x+2}{x-2}}-\sqrt{\dfrac{x-2}{x+2}}=4$.

35. $\dfrac{x+\sqrt{x^2+1}}{x-\sqrt{x^2+1}}+\dfrac{x-\sqrt{x^2+1}}{x+\sqrt{x^2+1}}=14$.

36. $\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\sqrt{x}}}}}=2$.

37. $1+\sqrt{1+x}=\sqrt{2+x+\sqrt{1+x}}$.

VIII. QUADRATIC EQUATIONS.

214. A **Quadratic Equation** is an equation of the second degree.

Thus, $ax^2 = b$, and $ax^2 + bx = c$.

215. There are two classes of quadratic equations, (1) *Pure*, or *Incomplete*, quadratics and (2) *Affected*, or *Complete*, quadratics.

216. A **Pure Quadratic Equation** is one which contains only the second power of the unknown quantity; as, $ax^2 = b$.

217. An **Affected Quadratic Equation** is one which contains both the second and first powers of the unknown quantity; as, $ax^2 + bx = c$.

PURE QUADRATIC EQUATIONS.

218. The **General Form** of a pure quadratic equation is

$$ax^2 = b.$$

1. Solve the equation $3x^2 - 5 = 22$.

Transposing, $3x^2 = 27$

Or, $x^2 = 9$

Extracting the square root, $x = \pm 3$, *Ans.*

2. Solve the equation $4x^2 + \dfrac{x^2}{5} = 105$.

Clearing of fractions, $20x^2 + x^2 = 525$

Uniting, $21x^2 = 525$

Or, $x^2 = 25$

Extracting the square root, $x = \pm 5$, *Ans.*

EXAMPLES.

Solve the following equations:

3. $x^2 - 5 = 59$.

4. $2x^2 - 5 = x^2 + 4$.

5. $5x^2 + 2 = 3x^2 + 34$.

6. $(x + 5)(x - 5) = 24$.

Find the values of—

2. $\sqrt{-4} + \sqrt{-9}$.
3. $\sqrt{-36} + \sqrt{-100}$.
4. $\sqrt{-121} - \sqrt{-81}$.
5. $\sqrt{-a^2} - \sqrt{-b^2}$.
6. $\sqrt{-49m^2} - \sqrt{-25m^2}$.
7. $\sqrt{-300} - \sqrt{-108}$.
8. $\sqrt{-32a^2} + a\sqrt{-8}$.
9. $\sqrt{-18b^2} - c\sqrt{-50}$.
10. $\sqrt{-(a+b)^2} + \sqrt{-(a-b)^2}$.
11. $\sqrt{-m^2 + 2mn - n^2} + \sqrt{-(m+n)^2}$.

206. To multiply imaginary quantities.

PRINCIPLE.

The **product** *of two imaginary quantities is real, with the sign before the result the* **reverse** *of that obtained by the common rule for multiplication.*

Thus, $\sqrt{-a^2} \times \sqrt{-b^2} = a\sqrt{-1} \times b\sqrt{-1} = ab \times -1 = -ab$

$-\sqrt{-a^2} \times \sqrt{-b^2} = -a\sqrt{-1} \times b\sqrt{-1} = -ab \times -1 = ab.$

1. Multiply $2\sqrt{-2}$ by $3\sqrt{-3}$.

By **Art.** 190,

$$\begin{array}{r} 2\sqrt{-2} = 2\sqrt{2} \times \sqrt{-1} \\ 3\sqrt{-3} = 3\sqrt{3} \times \sqrt{-1} \\ \hline 6\sqrt{6} \times -1 = -6\sqrt{6}, \textit{ Ans.} \end{array}$$

Multiply—

2. $\sqrt{-2}$ by $\sqrt{-3}$.
3. $\sqrt{-3}$ by $-\sqrt{-15}$.
4. $-\sqrt{-18}$ by $-\sqrt{-2}$.
5. $-\sqrt{-5}$ by $\sqrt{-3}$.
6. $2\sqrt{-3}$ by $3\sqrt{-12}$.
7. $1 + \sqrt{-1}$ by $1 + \sqrt{-1}$.
8. $1 - \sqrt{-1}$ by $1 - \sqrt{-1}$.
9. $1 + \sqrt{-1}$ by $1 - \sqrt{-1}$.
10. $\sqrt{a} + \sqrt{-b}$ by $\sqrt{a} - \sqrt{-b}$.
11. $a + b\sqrt{-1}$ by $a - b\sqrt{-1}$.
12. $\sqrt{-2}$ by $\sqrt{-3} \times \sqrt{-4}$.
13. $(\sqrt{a} + \sqrt{-b})(\sqrt{-a} + \sqrt{b})$ by $-\sqrt{-a^2}$.
14. $2 + 3\sqrt{-3}$ by $3 - 2\sqrt{-3}$.

Expand the following:

15. $(1-\sqrt{-2})^2$.

16. $(\sqrt{-3}-\sqrt{-2})^2$.

17. $(\sqrt{-m}-\sqrt{-n})^2$.

18. $(\sqrt{-3}+\sqrt{2})^2$.

19. $(a-\sqrt{-b})^3$.

20. $(\sqrt{-a}+b)^3$.

21. $(\sqrt{-a}-\sqrt{-b})^3$.

22. $(\sqrt{2}+\sqrt{-3})^3$.

207. To divide imaginary quantities.

PRINCIPLE.

The **quotient** *of two imaginary quantities is real, with the sign before the result the* **same** *as that obtained by the common rule for division.*

Thus, $$\frac{\sqrt{-ab}}{\sqrt{-b}}=\frac{\sqrt{ab}\times\sqrt{-1}}{\sqrt{b}\times\sqrt{-1}}=\sqrt{a}.$$

$$\frac{-\sqrt{-ab}}{\sqrt{-b}}=\frac{-\sqrt{ab}\times\sqrt{-1}}{\sqrt{b}\times\sqrt{-1}}=-\sqrt{a}.$$

1. Divide $10\sqrt{-6}$ by $5\sqrt{-2}$.

$$\frac{10\sqrt{-6}}{5\sqrt{-2}}=\frac{10\sqrt{6}\times\sqrt{-1}}{5\sqrt{2}\times\sqrt{-1}}=2\sqrt{3}.$$

2. Divide $2\sqrt{-1}$ by $1+\sqrt{-1}$.

$$2\sqrt{-1}\div 1+\sqrt{-1}=\frac{2\sqrt{-1}}{1+\sqrt{-1}}.$$

Rationalizing the denominator,

$$\frac{2\sqrt{-1}\,(1-\sqrt{-1})}{(1+\sqrt{-1})\,(1-\sqrt{-1})}=\frac{2\sqrt{-1}+2}{2}=1+\sqrt{-1}.$$

Divide—

3. $3\sqrt{-10}$ by $\sqrt{-2}$.

4. $8\sqrt{-12}$ by $\sqrt{-3}$.

5. $a\sqrt{-b^2}$ by $-\sqrt{-b}$.

6. $\sqrt[4]{-8}$ *by* $\sqrt[4]{-2}$.

7. $\sqrt[6]{-36}$ by $\sqrt[6]{-4}$.

8. $\sqrt[8]{-48}$ by $\sqrt[8]{-3}$.

9. 2 by $1-\sqrt{-1}$.

10. $-2\sqrt{-1}$ by $1-\sqrt{-1}$.

11. $\sqrt{15}$ by $\sqrt{-3}$.
12. $\sqrt{18}$ by $\sqrt{-2}$.
13. $a+b$ by $\sqrt{a}-\sqrt{-b}$.
14. 1 by $\sqrt{-2}+\sqrt{-3}$.
15. $24+5\sqrt{-3}$ by $2+3\sqrt{-3}$.
16. $9+\sqrt{15}$ by $\sqrt{-3}+\sqrt{-5}$.

PROPERTIES OF QUADRATIC SURDS.

208. A **Quadratic Surd** is the indicated square root of an imperfect square, as $\sqrt{a}$, $\sqrt{5}$, or $\sqrt{a+b}$.

PRINCIPLE.

A quadratic surd cannot equal the sum of a rational quantity and a quadratic surd.

If possible, let $\sqrt{a} = b + \sqrt{c}$

Squaring, $a = b^2 + 2b\sqrt{c} + c$

Or, $2b\sqrt{c} = a - b^2 - c$

And, $\sqrt{c} = \dfrac{a - b^2 - c}{2b}$

That is, a surd is equal to a rational quantity, which is impossible. Hence $\sqrt{a}$ cannot equal $b + \sqrt{c}$.

209. *If* $a + \sqrt{b} = c + \sqrt{d}$, *then* $a = c$ *and* $\sqrt{b} = \sqrt{d}$.

If a does not equal c, let $a = c \pm x$,

Substituting, $c \pm x + \sqrt{b} = c + \sqrt{d}$

Or, $\pm x + \sqrt{b} = \sqrt{d}$,

which is impossible by Art. 208, Prin.

Hence, $a = c$, and $\sqrt{b} = \sqrt{d}$.

210. *If* $\sqrt{x} + \sqrt{y} = \sqrt{a + \sqrt{b}}$, *then* $\sqrt{x} - \sqrt{y} = \sqrt{a - \sqrt{b}}$.

Squaring $\sqrt{x} + \sqrt{y} = \sqrt{a + \sqrt{b}}$, we have,

$x + 2\sqrt{xy} + y = a + \sqrt{b}$,

By Art. 209, $x + y = a$,

And, $2\sqrt{xy} = \sqrt{b}$

Subtracting, $x - 2\sqrt{xy} + y = a - \sqrt{b}$

Extracting the square root, $\sqrt{x} - \sqrt{y} = \sqrt{a - \sqrt{b}}$.

SQUARE ROOT OF A BINOMIAL SURD.

211. The preceding principles furnish an easy method of extracting the square root of binomial surds of the form

$$a + \sqrt{b}.$$

1. Extract the square root of $7 + 4\sqrt{3}$.

Let $\sqrt{x} + \sqrt{y} = \sqrt{7 + 4\sqrt{3}}$ (1)

Then, by Art. 210, $\sqrt{x} - \sqrt{y} = \sqrt{7 - 4\sqrt{3}}$ (2)

Multiplying, $x - y = \sqrt{49 - 48}$ (3)

Or, $x - y = 1$ (4)

Squaring (1), $x + 2\sqrt{xy} + y = 7 + 4\sqrt{3}$ (5)

By Art. 209, $x + y = 7$ (6)

Adding (4) and (6), $2x = 8$

$x = 4$

Subtracting (4) from (6), $2y = 6$

$y = 3$

Substituting in (1), $\sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}$, *Ans.*

212. Examples like the above may be solved by inspection, by expressing them in the form

$$x + 2\sqrt{xy} + y,$$

in which the coefficient of the surd term is 2, and the rational term is separated into two parts whose product is equal to the quantity under the radical sign. Then extract the square root of these parts, and connect them by the sign of the surd.

1. Extract the square root of $14 + 6\sqrt{5}$.

$$\sqrt{14 + 6\sqrt{5}} = \sqrt{14 + 2\sqrt{45}}$$

We separate 14 into two parts whose product is 45. These parts are 9 and 5; hence,

$$\sqrt{14 + 2\sqrt{45}} = \sqrt{9 + 2\sqrt{45} + 5}$$
$$= 3 + \sqrt{5}, \textit{ Ans.}$$

2. Extract the square root of $28 + 10\sqrt{3}$.

$$\sqrt{28 + 10\sqrt{3}} = \sqrt{28 + 2\sqrt{75}}$$

We separate 28 into two parts whose product is 75. These parts are 25 and 3; hence,

$$\sqrt{28 + 2\sqrt{75}} = \sqrt{25 + 2\sqrt{75} + 3}$$
$$= 5 + \sqrt{3}, \textit{ Ans.}$$

EXAMPLES.

Extract the square root of—

3. $11 + 4\sqrt{7}$.
4. $10 + 4\sqrt{6}$.
5. $14 + 4\sqrt{10}$.
6. $11 - 6\sqrt{2}$.
7. $29 + 12\sqrt{5}$.
8. $14 - \sqrt{180}$.
9. $5 + \sqrt{24}$.
10. $30 - 12\sqrt{6}$.
11. $1 + 4\sqrt{-3}$.
12. $16 + \sqrt{252}$.
13. $28 - 4\sqrt{48}$.
14. $43 - \sqrt{1800}$.
15. $120 - \sqrt{8000}$.
16. $12 - \sqrt{108}$.
17. $47 - 6\sqrt{10}$.
18. $28 - \sqrt{300}$.
19. $7 - 6\sqrt{-2}$.
20. $16 + 4\sqrt{7}$.
21. $19 - 6\sqrt{10}$.
22. $+\sqrt{-1}$.
23. $\sqrt{18} - 4$.
24. $3\sqrt{6} + 2\sqrt{12}$.
25. $4\sqrt{6} + 6\sqrt{2}$.
26. $3\sqrt{5} + 2\sqrt{10}$.
27. $3\sqrt{3} - 2\sqrt{6}$.
28. $6\sqrt{-1}$.
29. $-4\sqrt{-1}$.
30. $a^2 - 2x\sqrt{a^2 - x^2}$.
31. $b + 2a\sqrt{b - a^2}$.
32. $2m - 2\sqrt{m^2 - x^2}$.
33. $6 - m - 4\sqrt{2 - m}$.

RADICAL EQUATIONS.

213. Radical Equations are those which contain the unknown quantity in the form of a radical.

A radical equation is solved by rationalizing the terms containing the unknown quantity, and then determining its value.

1. Solve the equation $\sqrt{x} + 5 = 14$.

Transposing, $\sqrt{x} = 9$

Squaring, $x = 81$, Ans.

2. Solve the equation $\sqrt{2x+5}+\sqrt{2x-3}=4$.

Transposing, $\sqrt{2x+5} = 4 - \sqrt{2x-3}$

Squaring, $2x+5 = 16 - 8\sqrt{2x-3} + 2x - 3$

Transposing and uniting, $8\sqrt{2x-3} = -8$

Or, $\sqrt{2x-3} = -1$

Squaring, $2x - 3 = 1$

$2x = 4$

$x = 2$, *Ans.*

3. Solve the equation $\dfrac{\sqrt{2x+2}+\sqrt{2x-2}}{\sqrt{2x+2}-\sqrt{2x-2}}=4$.

Rationalizing the denominator, we have

$$\frac{4x+2\sqrt{4x^2-4}}{4}=4$$

Or, $4x+2\sqrt{4x^2-4}=16$

Transposing $4x$, and dividing by 2,

$$\sqrt{4x^2-4} = 8 - 2x$$

Squaring, $4x^2 - 4 = 64 - 32x + 4x^2$

Collecting, $32x = 68$

$x = 2\frac{1}{8}$, *Ans.*

EXAMPLES.

Solve the following equations:

4. $\sqrt{3x}+2=8$.
5. $\sqrt{3x+4}+1=5$.
6. $\sqrt{2x+2}-5=1$.
7. $6=\sqrt[3]{4x}+4$.
8. $\sqrt{2x+5}=\sqrt{2x}+1$.
9. $\sqrt{4x^2-3}=2x-3$.
10. $\sqrt{9x^2-20}=3x-2$.
11. $5x-2=\sqrt{25x^2-36}$.
12. $\sqrt{x+a}=\dfrac{3a}{\sqrt{x+a}}$.
13. $2\sqrt{x}+3=\sqrt{4x+11}$.
14. $\sqrt{x^2-5x+10}=x-1$.
15. $\sqrt{x-4}+\sqrt{x+9}=13$.
16. $\sqrt{x-11}+\sqrt{x+4}=15$.
17. $\sqrt{x-20}=10-\sqrt{x}$.
18. $\sqrt{x-8}=6-\sqrt{x+4}$.
19. $\sqrt{x^2+4\sqrt{x^2+13}}=x+2$.
20. $\sqrt{m^2+x}+\sqrt{m^2-x}=2n$.
21. $\dfrac{x+a}{2\sqrt{x}}=\dfrac{\sqrt{4x}}{3}$.

22. $\sqrt{x+5}+\sqrt{x-5}=\sqrt{2x+4}.$

23. $\dfrac{\sqrt{x}+5}{\sqrt{x}+6}=\dfrac{\sqrt{x}-7}{\sqrt{x}-8}.$

24. $x+\sqrt{2-x}=\dfrac{2}{\sqrt{2-x}}.$

25. $\dfrac{\sqrt{x}+25}{\sqrt{x}+7}=\dfrac{\sqrt{x}+4}{\sqrt{x}+1}.$

26. $\sqrt{2x}+\sqrt{2x+9}=\dfrac{45}{\sqrt{2x+9}}.$

27. $\sqrt{x}-\sqrt{x-8}=\dfrac{6}{\sqrt{x}}.$

28. $\dfrac{1}{\sqrt{x-1}-1}+\dfrac{1}{\sqrt{x-1}+1}=\dfrac{3}{\sqrt{x-1}}.$

29. $\dfrac{x-4}{2+\sqrt{x}}=2\sqrt{x}-9+\dfrac{4\sqrt{x}}{3}.$

30. $\dfrac{n+x}{\sqrt{n+x}+\sqrt{n}}+\dfrac{n+x}{\sqrt{n+x}-\sqrt{n}}=4\sqrt{n}+x.$

31. $x-6=\dfrac{x^2}{(1+\sqrt{1+x})^2}.$

32. $\sqrt{2-\sqrt{4-x^2}}=\dfrac{\sqrt{5}-1}{2}.$

33. $\dfrac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=m.$

34. $\sqrt{\dfrac{x+2}{x-2}}-\sqrt{\dfrac{x-2}{x+2}}=4.$

35. $\dfrac{x+\sqrt{x^2+1}}{x-\sqrt{x^2+1}}+\dfrac{x-\sqrt{x^2+1}}{x+\sqrt{x^2+1}}=14.$

36. $\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\sqrt{x}}}}}=2.$

37. $1+\sqrt{1+x}=\sqrt{2+x+\sqrt{1+x}}.$

VIII. QUADRATIC EQUATIONS.

214. A **Quadratic Equation** is an equation of the second degree.

Thus, $ax^2 = b$, and $ax^2 + bx = c$.

215. There are two classes of quadratic equations, (1) *Pure*, or *Incomplete*, quadratics and (2) *Affected*, or *Complete*, quadratics.

216. A **Pure Quadratic Equation** is one which contains only the second power of the unknown quantity; as, $ax^2 = b$.

217. An **Affected Quadratic Equation** is one which contains both the second and first powers of the unknown quantity; as, $ax^2 + bx = c$.

PURE QUADRATIC EQUATIONS.

218. The **General Form** of a pure quadratic equation is

$$ax^2 = b.$$

1. Solve the equation $3x^2 - 5 = 22$.

Transposing, $3x^2 = 27$

Or, $x^2 = 9$

Extracting the square root, $x = \pm 3$, *Ans.*

2. Solve the equation $4x^2 + \frac{x^2}{5} = 105$.

Clearing of fractions, $20x^2 + x^2 = 525$

Uniting, $21x^2 = 525$

Or, $x^2 = 25$

Extracting the square root, $x = \pm 5$, *Ans.*

EXAMPLES.

Solve the following equations:

3. $x^2 - 5 = 59$.

4. $2x^2 - 5 = x^2 + 4$.

5. $5x^2 + 2 = 3x^2 + 34$.

6. $(x + 5)(x - 5) = 24$.

7. $2x + 6x^{-1} = 4x - 12x^{-1}$.

8. $\frac{x^2 - 10}{3} = \frac{2x^2 - 10}{8}$.

9. $\frac{x^2}{3} + 4 = \frac{3x^2 - 1}{6}$.

10. $\frac{3}{2 - x} + \frac{3}{2 + x} = 4$.

11. $\frac{x^2 + 15x}{6} = \frac{5(x + 2)}{2}$.

12. $\sqrt{x^2 + 44} = 4\sqrt{x^2 - 1}$.

13. $(x + 2)^3 + (x - 2)^3 = 56x$.

14. $\frac{x + 8}{4} = \frac{9}{x - 8}$.

15. $\frac{x + 5}{x - 5} + \frac{x - 5}{x + 5} = 6\frac{1}{6}$.

16. $\sqrt{x - n} = \frac{a}{\sqrt{x - n}}$.

17. $\sqrt{x + 3} = \frac{4}{\sqrt{x - 3}}$.

18. $\sqrt{2x^2 - 9} = \sqrt{x^2 + 16}$.

19. $(x + 5)^2 + 4 = 10x + 38$.

20. $(x + 4)^2 + (x - 2)^2 = 92 + 4x$.

21. $(x - 2)(x + 3) + (x + 4)(x - 2) = 274 + 3x$.

22. $(x + 3)^2 + (x - 5)(x + 4) = 61 + 5x$.

23. $(x + 6)^2 + 32 = (x - 8)^2 + x^2 + 28x$.

24. $x + \sqrt{x^2 - \sqrt{17 - 4x}} = 1$.

25. $\sqrt{x + 4} = \sqrt{x + \sqrt{x^2 - 9}}$.

26. $\sqrt{x + n} = \sqrt[4]{x^2 + a^2}$.

27. $\frac{3}{x + \sqrt{3 + x^2}} + \frac{3}{x - \sqrt{3 + x^2}} + \frac{8}{x} = 0$.

28. $\frac{4x}{1 + x + \sqrt{1 + x^2}} + \frac{4x}{1 + x - \sqrt{1 + x^2}} = \frac{72}{x} + 4$.

29. $\sqrt{x + \sqrt{x^2 - 4}} = \sqrt{a + x}$.

AFFECTED QUADRATICS.

219. An **Affected Quadratic Equation** is solved by adding such a quantity to both members of the equation as will make the *first member* a perfect square. This process is called *Completing the Square.*

There are several methods of *completing the square.* The two in general use are as follows:

FIRST METHOD OF COMPLETING THE SQUARE.

220. Every affected quadratic equation can be reduced to the form

$$x^2 + 2px = q.$$

1. Solve the equation $x^2 + 6x = 16$.

Since *the square of a binomial is equal to the square of the first term, plus twice the product of the first term into the second, plus the square of the second,* (ART. 86, Prin. 1), if we regard $x^2 + 6x$ as the first two terms of the square of a binomial, the first term of the binomial will be the square root of x^2, or x; and $6x$ will be twice the first term into the second; hence, the second term is $6x$ divided by $2x$ or 3. Adding 3^2 or 9 to the first member, to complete the square, and to the second member to preserve the equality, we have

$$x^2 + 6x + 9 = 25 \qquad (1)$$

Extracting the square root of both members,

$$x + 3 = \pm 5 \qquad (2)$$

Transposing, $x = -3 \pm 5 = 2$ or -8, *Ans.*

Therefore, to solve an affected quadratic, we have the following

RULE.

Reduce the equation to the form $x^2 + 2px = q$.

Add to both members the square of half the coefficient of x.

Extract the square root of both members, and solve the resulting simple equation.

2. Solve the equation $x^2 + 10x = 24$.

Completing the square by adding 5^2,

$$x^2 + 10x + 25 = 49$$

Extracting the square root, $x + 5 = \pm 7$

$x = -5 \pm 7 = 2$, or -12, Ans.

3. Solve the equation $x^2 - 2(a-b)x = 4ab$.

Completing the square by adding $(a-b)^2$ to both members,

$$x^2 + 2(a-b)x + (a-b)^2 = a^2 + 2ab + b^2$$

Extracting the square root,

$$x + (a-b) = \pm(a+b)$$

$$x = -(a-b) \pm (a+b) = 2b, \text{ or } -2a, \textit{ Ans.}$$

EXAMPLES.

Solve the following equations:

4. $x^2 + 6x = 16$.
5. $x^2 + 8x = 9$.
6. $x^2 + 10x = 11$.
7. $x^2 + 12x = 13$.
8. $x^2 - 14x = 51$.
9. $x^2 - 16x = 17$.
10. $x^2 - 18x = 19$.
11. $x^2 - 20x = 300$.
12. $x^2 + 30x = -29$.
13. $x^2 + 22x = -21$.
14. $x^2 + 24x = -25$.
15. $x^2 + 28x = -52$.
16. $x^2 - 10x = -9$.
17. $x^2 - 40x = -175$.
18. $x^2 - 60x = -500$.
19. $x^2 - 14x = -24$.
20. $x^2 + 100x = 204$.
21. $x^2 - 120x = 1584$.

FIRST METHOD OF WRITING THE RESULT BY FORMULA.

221. In the equation $x^2 + 2px = q$, to which form every affected quadratic can be reduced (ART. 220),

the square is completed by adding p^2 to both members,

$$x^2 + 2px + p^2 = q + p^2$$

Extracting the square root, $x + p = \pm\sqrt{q + p^2}$

$$x = -p \pm \sqrt{q + p^2} \qquad (1)$$

Using this result as a *formula*, the roots of an affected quadratic can be written without completing the square. Thus,

1. Solve the equation, $x^2 + 10x = 56$.

In this case, $p = 5$, and $q = 56$; substituting these values in (1),

$$x = -5 \pm \sqrt{56 + 25} = 4, \text{ or } -14, \text{ Ans.}$$

2. Solve the equation $x^2 - 8x = 20$.

In this case, $p = 4$, and $q = 20$; therefore,

$$x = 4 \pm \sqrt{20 + 16} = 10 \text{ or } -2, \textit{ Ans.}$$

3. Solve the equation $x^2 - 2ax = b$.

In this case, $p = a$, and $q = b$; therefore,

$$x = a \pm \sqrt{b + a^2}, \textit{ Ans.}$$

EXAMPLES.

Write the roots to the following equations:

4. $x^2 - 4x = 21$.
5. $x^2 + 12x = -11$.
6. $x^2 - 14x = -13$.
7. $x^2 + 4x = 1$.
8. $x^2 - 2px = -q$.
9. $x^2 + 2nx = m^2 - n^2$.
10. $x^2 + 2ax + a^2 = 0$.
11. $x^2 + 2ax = b^2 + a^2$.
12. $x - 40x^{-1} = -6$.
13. $x - fx^{-1} - 2n = 0$.
14. $5x^2 + 40x - 45 = 0$.
15. $7x^2 - 28x = 7$.
16. $ax^2 + 4a^2x + ab - 5a^2 = 0$.
17. $(x + 3)^2 + 2ax = b$.
18. $x^2 + 2(a - b)x = 4ab$.
19. $x^2 + 2(n + 1)x = -4n$.
20. $x^2 + 2(m - n)x = 4mn + 4n$.
21. $x^2 - 2(a + b)x = 3a^2 - b^2 - 2ab$.

SECOND METHOD OF COMPLETING THE SQUARE.

222. If a quadratic of the form

$$ax^2 + bx = c$$

be solved by the preceding method, it will involve fractions, which will make the process long and difficult. The following method is used to avoid fractions:

1. Solve the equation $ax^2 + bx = c$.

To complete the square without using fractions, the first term must be a perfect square, and the second term must be divisible by 2. To accomplish this we multiply both members by $4a$, which gives

$$4a^2x^2 + 4abx = 4ac \qquad (1)$$

Completing the square by ART. 220,

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2 \qquad (2)$$

Extracting the square root, $2ax + b = \pm\sqrt{4ac + b^2}$ (3)

Transposing, $2ax = -b \pm\sqrt{4ac + b^2}$

And, $x = \dfrac{-b \pm\sqrt{4ac + b^2}}{2a}$, *Ans.*

Therefore, to solve an affected quadratic equation without using fractions,

RULE.

Reduce the equation to the form $ax^2 + bx = c$.

Multiply the equation by 4 *times the coefficient of* x^2, *and add the square of the coefficient of* x *to both members.*

Extract the square root, and find the value of x *in the resulting equation.*

2. Solve the equation $2x^2 + 3x = 5$.

Multiplying both members by 4 times 2, or 8,

$$16x^2 + 24x = 40,$$

Adding to each member the square of 3,

$$16x^2 + 24x + 9 = 49,$$

Extracting the square root, $4x + 3 = \pm 7$,

Transposing, $4x = -3 \pm 7 = 4$, or -10

$x = 1$, or $-2\frac{1}{2}$, *Ans.*

EXAMPLES.

Solve the following equations by completing the square:

3. $2x^2 - 3x = 2$.
4. $3x^2 - 2x = 21$.
5. $4x^2 + x = 18$.
6. $2x^2 - 7x = -6$.
7. $5x^2 - 3x = 68$.
8. $3x^2 - 20x = -25$.
9. $4x^2 - x = 150$.
10. $2x^2 + 7x = 22$.
11. $x^2 + x = 20$.
12. $x^2 + 9x = -20$.
13. $6x^2 + 30x = -36$.
14. $3x^2 + 18x = -15$.
15. $6x^2 - 3x = 45$.
16. $3x^2 + 8x = 28$.

17. $2x^2 + 7x = 9$.

18. $ax^2 + dx = m$.

19. $mx^2 + mx = p$.

20. $x^2 + (n + 1)x = m$.

21. $(x - a)^2 + b = a^2$.

22. $x^2 - (a + b)x = -ab$.

23. $2x^2 + 3x = 13$.

24. $5x^2 + 7x = 9$.

25. $3x^2 + 2x = 15$.

26. $2x^2 + 5x = 11$.

SECOND METHOD OF WRITING THE RESULT BY FORMULA.

223. It was shown in Art. 222 that if $ax^2 + bx = c$, then

$$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}. \qquad (1)$$

Using this result as a *formula*, the roots of a quadratic of the form $ax^2 + bx = c$ can be written without completing the square. Thus,

1. Solve the equation $2x^2 - 3x = 2$.

In this case, $a = 2$, $b = -3$, and $c = 2$; substituting these values in (1),

$$x = \frac{3 \pm \sqrt{16 + 9}}{4} = 2, \text{ or } -\tfrac{1}{2}, \textit{ Ans.}$$

2. Solve the equation $5x^2 + x = 18$.

In this case, $a = 5$, $b = 1$, and $c = 18$; therefore,

$$x = \frac{-1 \pm \sqrt{360 + 1}}{10} = 1\tfrac{4}{5}, \text{ or } -2, \textit{ Ans.}$$

EXAMPLES.

Write the roots to the following equations:

3. $2x^2 - 6x = 20$.

4. $3x^2 - 10x = -3$.

5. $5x^2 - 6x = 8$.

6. $x^2 + 3x = 4$.

7. $x^2 + 3x = 5$.

8. $x^2 - 3x = 5$.

9. $2x^2 - 7x = 4$.

10. $mx^2 + nx = p$.

11. $x^2 - 3(a + b)x = -9ab$.

12. $(x + 3)^2 + 10(x - 1)^2 = 35$.

13. $x^2 - (2x - 3)^2 = -24$.

14. $(x + a)^2 - (x - a)^2 = b$.

15. $(a+b)x^2+3(a+b)x=a^2-b^2.$

16. $x^2-(m-n+p)x=(n-m)p.$

17. $(x+4)(x-4)-(8x-3)=-25.$

18. $\dfrac{5}{x}+\dfrac{x}{5}=6.$

19. $\dfrac{x^2-3}{x}+\dfrac{x^2+3}{2x}=4.$

20. $x+\dfrac{b}{cx}=a.$

21. $\dfrac{2}{x+1}+\dfrac{3}{x+2}=\dfrac{4}{x+3}.$

22. $\dfrac{x}{x-1}-\dfrac{x}{x-2}=\dfrac{3}{x-3}.$

23. $\dfrac{x}{8-x}+\dfrac{3x}{x-5}=21.$

24. $5x-\dfrac{24}{x}=4x+2.$

25. $\dfrac{x}{x-7}+\dfrac{x+7}{4}=8\frac{1}{2}.$

26. $\dfrac{1}{x-1}-\dfrac{2}{x-2}=\dfrac{3}{x-3}.$

27. $\dfrac{x}{x-a}-\dfrac{x+a}{x}=b.$

28. $\dfrac{2x+a}{3}+\dfrac{x^2-a^2}{x}=-\dfrac{5a}{x}.$

29. $\sqrt{x}+\dfrac{1}{\sqrt{x}}=2\frac{1}{2}.$

MISCELLANEOUS EXAMPLES.

224. The results to the following examples may be written by either of the preceding methods. The student should select the method best adapted to each equation.

1. $4x^2-3x=x^2+60.$

2. $-x^2+x=-30.$

3. $x-15x^{-1}=-2.$

4. $x^2-4x-21=0.$

5. $ax^2-a^2x=1.$

6. $(x+a)^3-(x-a)^3=3a^3.$

7. $(x+3)^3-(x-3)^3=504.$

8. $x^3-(x+a)^3=b.$

9. $5x+2=3x^2.$

10. $\sqrt{2x+15}+\sqrt{x-4}=6.$

11. $(n^2+1)x=nx^2+n.$

12. $(ax-1)^2+(x+a)^2=(a^4-1)x.$

13. $\sqrt{x}+1=\dfrac{x^2}{\sqrt{x}-1}.$

14. $\sqrt{2x+13}+\sqrt{3x-2}=9.$

15. $\dfrac{x^2-1}{x^2+1}=\dfrac{n+1}{n-1}.$

16. $\dfrac{x}{4-x}+\dfrac{4-x}{x}=3\tfrac{1}{3}.$

17. $\dfrac{x-4}{x}-\dfrac{x}{x+4}=-\dfrac{4}{15}.$

18. $x^2+2px=q.$

19. $x^2-2px=q.$

20. $x^2+2px=-q.$

21. $x^2-2px=-q.$

22. $(x-2)(x-3)=30.$

23. $(x-2)^3-(x-3)^3=19.$

24. $x^2+ax=b.$

25. $x^2-nx-mx=mn.$

26. $x^2-2px=q^3-p^2.$

27. $x^2+2mx=m^2.$

28. $5x^{-1}-2x=-9.$

29. $2x-4=70x^{-1}.$

30. $\dfrac{4}{5-x}+\dfrac{3}{4-x}=\dfrac{2}{3-x}.$

31. $\dfrac{9-x}{9+x}-\dfrac{9+x}{9-x}=-3\tfrac{8}{14}.$

32. $\dfrac{x-2}{x+3}+\dfrac{x-4}{x+5}=\dfrac{19}{40}.$

33. $\dfrac{x}{4}+\dfrac{5}{x}=\dfrac{x}{6}+\dfrac{7}{x}+\dfrac{5}{x}.$

34. $4\left(\dfrac{x-2}{x+2}\right)+\dfrac{x+2}{x-2}=\dfrac{13}{3}.$

35. $\dfrac{x+1}{x-1}=\dfrac{x^2-1}{9}.$

36. $\sqrt{3x+4}+\sqrt{2x+1}=\sqrt{12x+1}.$

37. $x^2-2(m-n+p)x=4(n-m)p.$

38. $x+\sqrt{1+x^2}=\dfrac{3}{\sqrt{1+x^2}}.$

39. $\sqrt{x^2-2x+1}=2(x-4).$

40. $(m+n)^2x^2-(m^2-n^2)x=mn.$

41. $(x-1)(x-2)(x-3)=x^3-58.$

EQUATIONS IN QUADRATIC FORM.

225. An equation is in the **Quadratic Form** when it contains only two powers of an unknown quantity, and the exponent of one power is twice that of the other; as,

$$x^4 + 2x^2 = 5.$$

$$x + 3x^{\frac{1}{2}} = 4.$$

$$(x - 3)^3 + (x - 3)^{\frac{3}{2}} = 5.$$

CASE I.

226. When the unknown term of the quadratic is a monomial.

1. Solve the equation $x^4 + 6x^2 = 40$.

Writing the result by ART. 221,

$$x^2 = -3 \pm \sqrt{40 + 9} = 4, \text{ or } -10,$$

And $$x = \pm 2, \text{ or } \pm \sqrt{-10}, \textit{ Ans.}$$

2. Solve the equation $x^{\frac{2}{3}} + x^{\frac{1}{3}} = 6$.

Writing the result by ART. 223,

$$x^{\frac{1}{3}} = \frac{-1 \pm \sqrt{24 + 1}}{2} = 2, \text{ or } -3,$$

And $$x = 8, \text{ or } -27, \textit{ Ans.}$$

3. Solve the equation $ax^{2n} + bx^n = c$.

Writing the result by ART. 223,

$$x^n = \frac{-b \pm \sqrt{4ac + b^2}}{2a},$$

And $$x = \left(\frac{-b \pm \sqrt{4ac + b^2}}{2a}\right)^{\frac{1}{n}}.$$

EXAMPLES.

Solve the following equations:

4. $x^4 - 8x^2 = 9$.

5. $x^6 - 5x^3 = -4$.

6. $3\sqrt[3]{x} + 2\sqrt[3]{x^2} = 2$.

7. $x^{2n} - 8x^n = 7$.

8. $x^6 - 7x^3 = 8$.

9. $x + \sqrt{x} = 20$.

10. $2x^4 - 3x^2 = 20$.

11. $2x^{\frac{1}{3}} + 5x^{\frac{2}{3}} = 24$.

12. $3x^{\frac{1}{4}} + 2x^{\frac{1}{2}} = 14$.

13. $2x^3 + 5x^{\frac{3}{2}} = 3$.

14. $ax^n - bx^{\frac{n}{2}} = c$.

15. $mx^{\frac{2}{3}} - mx^{\frac{1}{3}} = n^2$.

16. $x^8 - 9x^4 = 112$.

17. $x^{10} - 31x^5 = 32$.

18. $2x^2 + 3x^{-2} = 18\frac{1}{8}$.

19. $x^{\frac{1}{5}} + \sqrt[5]{x^2} = 12$.

20. $\dfrac{3\sqrt{x} + 7}{\sqrt{x} + 3} = \dfrac{\sqrt{x} + 5}{\sqrt{x}}$.

21. $x - 1 = \dfrac{2\sqrt{x} - 2}{x^{\frac{1}{2}}}$.

22. $x - 4 = \dfrac{3x + 6\sqrt{x}}{x}$.

23. $\dfrac{x - \sqrt{x^2 - 4}}{x + \sqrt{x^2 - 4}} = (x - 4)^2$.

24. $\dfrac{x}{\sqrt{x} + \sqrt{x - 5}} + \dfrac{x}{\sqrt{x} - \sqrt{x - 5}} = \dfrac{10}{\sqrt{x}}$.

25. $\sqrt{3x + 1} + \sqrt{2x - 1} = \sqrt{9x + 4}$.

CASE II.

227. When the unknown term of the quadratic is a polynomial.

In this case the polynomial may be regarded as a single quantity, and solved with reference to that quantity.

1. Solve the equation $(x - 2)^2 + 6(x - 2) = 7$.

Writing the result by ART. 221,

$$x - 2 = -3 \pm \sqrt{7 + 9} = 1, \text{ or } -7,$$

And, $x = 3$, or -5, *Ans.*

EQUATIONS IN QUADRATIC FORM.

225. An equation is in the **Quadratic Form** when it contains only two powers of an unknown quantity, and the exponent of one power is twice that of the other; as,

$$x^4 + 2x^2 = 5.$$

$$x + 3x^{\frac{1}{2}} = 4.$$

$$(x - 3)^3 + (x - 3)^{\frac{3}{2}} = 5.$$

CASE I.

226. When the unknown term of the quadratic is a monomial.

1. Solve the equation $x^4 + 6x^2 = 40$.

Writing the result by Art. 221,

$$x^2 = -3 \pm \sqrt{40 + 9} = 4, \text{ or } -10,$$

And $x = \pm 2, \text{ or } \pm \sqrt{-10}$, *Ans.*

2. Solve the equation $x^{\frac{2}{3}} + x^{\frac{1}{3}} = 6$.

Writing the result by Art. 223,

$$x^{\frac{1}{3}} = \frac{-1 \pm \sqrt{24 + 1}}{2} = 2, \text{ or } -3,$$

And $x = 8, \text{ or } -27$, *Ans.*

3. Solve the equation $ax^{2n} + bx^n = c$.

Writing the result by Art. 223,

$$x^n = \frac{-b \pm \sqrt{4ac + b^2}}{2a},$$

And

$$x = \left(\frac{-b \pm \sqrt{4ac + b^2}}{2a}\right)^{\frac{1}{n}}.$$

EXAMPLES.

Solve the following equations:

4. $x^4 - 8x^2 = 9$.

5. $x^6 - 5x^3 = -4$.

6. $3\sqrt[3]{x} + 2\sqrt[3]{x^2} = 2$.

7. $x^{2n} - 8x^n = 7$.

8. $x^6 - 7x^3 = 8.$

9. $x + \sqrt{x} = 20.$

10. $2x^4 - 3x^2 = 20.$

11. $2x^{\frac{1}{3}} + 5x^{\frac{2}{3}} = 24.$

12. $3x^{\frac{1}{4}} + 2x^{\frac{1}{2}} = 14.$

13. $2x^3 + 5x^{\frac{3}{2}} = 3.$

14. $ax^n - bx^{\frac{n}{2}} = c.$

15. $mx^{\frac{2}{3}} - mx^{\frac{1}{3}} = n^2.$

16. $x^8 - 9x^4 = 112.$

17. $x^{10} - 31x^5 = 32.$

18. $2x^2 + 3x^{-2} = 18\frac{1}{8}.$

19. $x^{\frac{1}{5}} + \sqrt[5]{x^2} = 12.$

20. $\dfrac{3\sqrt{x} + 7}{\sqrt{x} + 3} = \dfrac{\sqrt{x} + 5}{\sqrt{x}}.$

21. $x - 1 = \dfrac{2\sqrt{x} - 2}{x^{\frac{1}{2}}}.$

22. $x - 4 = \dfrac{3x + 6\sqrt{x}}{x}.$

23. $\dfrac{x - \sqrt{x^2 - 4}}{x + \sqrt{x^2 - 4}} = (x - 4)^2.$

24. $\dfrac{x}{\sqrt{x} + \sqrt{x - 5}} + \dfrac{x}{\sqrt{x} - \sqrt{x - 5}} = \dfrac{10}{\sqrt{x}}.$

25. $\sqrt{3x + 1} + \sqrt{2x - 1} = \sqrt{9x + 4}.$

CASE II.

227. When the unknown term of the quadratic is a polynomial.

In this case the polynomial may be regarded as a single quantity, and solved with reference to that quantity.

1. Solve the equation $(x - 2)^2 + 6(x - 2) = 7.$

Writing the result by ART. 221,

$$x - 2 = -3 \pm \sqrt{7 + 9} = 1, \text{ or } -7,$$

And,

$$x = 3, \text{ or } -5, \textit{ Ans.}$$

3. Solve the equation $x^2 + 2(a-b)x = 4ab$.

Completing the square by adding $(a-b)^2$ to both members,

$$x^2 + 2(a-b)x + (a-b)^2 = a^2 + 2ab + b^2$$

Extracting the square root,

$$x + (a-b) = \pm (a+b)$$

$$x = -(a-b) \pm (a+b) = 2b, \text{ or } -2a, \textit{Ans.}$$

EXAMPLES.

Solve the following equations:

4. $x^2 + 6x = 16$.
5. $x^2 + 8x = 9$.
6. $x^2 + 10x = 11$.
7. $x^2 + 12x = 13$.
8. $x^2 - 14x = 51$.
9. $x^2 - 16x = 17$.
10. $x^2 - 18x = 19$.
11. $x^2 - 20x = 300$.
12. $x^2 + 30x = -29$.
13. $x^2 + 22x = -21$.
14. $x^2 + 24x = -25$.
15. $x^2 + 28x = -52$.
16. $x^2 - 10x = -9$.
17. $x^2 - 40x = -175$.
18. $x^2 - 60x = -500$.
19. $x^2 - 14x = -24$.
20. $x^2 + 100x = 204$.
21. $x^2 - 120x = 1584$.

FIRST METHOD OF WRITING THE RESULT BY FORMULA.

221. In the equation $x^2 + 2px = q$, to which form every affected quadratic can be reduced (Art. 220),

the square is completed by adding p^2 to both members,

$$x^2 + 2px + p^2 = q + p^2$$

Extracting the square root, $x + p = \pm\sqrt{q + p^2}$

$$x = -p \pm \sqrt{q + p^2} \qquad (1)$$

Using this result as a *formula*, the roots of an affected quadratic can be written without completing the square. Thus,

1. Solve the equation, $x^2 + 10x = 56$.

In this case, $p = 5$, and $q = 56$; substituting these values in (1),

$$x = -5 \pm \sqrt{56 + 25} = 4, \text{ or } -14, \text{ Ans.}$$

2. Solve the equation $x^2 - 8x = 20$.

In this case, $p = 4$, and $q = 20$; therefore,

$$x = 4 \pm \sqrt{20 + 16} = 10 \text{ or } -2, \textit{ Ans.}$$

3. Solve the equation $x^2 - 2ax = b$.

In this case, $p = a$, and $q = b$; therefore,

$$x = a \pm \sqrt{b + a^2}, \textit{ Ans.}$$

EXAMPLES.

Write the roots to the following equations:

4. $x^2 - 4x = 21$.
5. $x^2 + 12x = -11$.
6. $x^2 - 14x = -13$.
7. $x^2 + 4x = 1$.
8. $x^2 - 2px = -q$.
9. $x^2 + 2nx = m^2 - n^2$.
10. $x^2 + 2ax + a^2 = 0$.
11. $x^2 + 2ax = b^2 + a^2$.
12. $x - 40x^{-1} = -6$.
13. $x - fx^{-1} - 2n = 0$.
14. $5x^2 + 40x - 45 = 0$.
15. $7x^2 - 28x = 7$.
16. $ax^2 + 4a^2x + ab - 5a^2 = 0$.
17. $(x + 3)^2 + 2ax = b$.
18. $x^2 + 2(a - b)x = 4ab$.
19. $x^2 + 2(n + 1)x = -4n$.
20. $x^2 + 2(m - n)x = 4mn + 4n$.
21. $x^2 - 2(a + b)x = 3a^2 - b^2 - 2ab$.

SECOND METHOD OF COMPLETING THE SQUARE.

222. If a quadratic of the form

$$ax^2 + bx = c$$

be solved by the preceding method, it will involve fractions, which will make the process long and difficult. The following method is used to avoid fractions:

1. Solve the equation $ax^2 + bx = c$.

To complete the square without using fractions, the first term must be a perfect square, and the second term must be divisible by 2. To accomplish this we multiply both members by $4a$, which gives

$$4a^2x^2 + 4abx = 4ac \qquad (2)$$

Adding the square of b,

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2 \qquad (3)$$

Extracting the square root, $2ax + b = \pm\sqrt{4ac + b^2}$ (4)

Transposing, $2ax = -b \pm \sqrt{4ac + b^2}$

Hence $x = \dfrac{-b \pm \sqrt{4ac + b^2}}{2a}$, *Ans.*

Therefore, to solve an affected quadratic equation without using fractions,

RULE.

Reduce the equation to the form $ax^2 + bx = c$.

Multiply the equation by 4 times the coefficient of x^2, *and add the square of the coefficient of* x *to both members.*

Extract the square root, and find the value of x *in the resulting equation.*

2. Solve the equation $2x^2 + 3x = 5$.

Multiplying both members by 4 times 2, or 8,

$$16x^2 + 24x = 40,$$

Adding to each member the square of 3,

$$16x^2 + 24x + 9 = 49,$$

Extracting the square root, $4x + 3 = \pm 7$,

Transposing, $4x = -3 \pm 7 = 4$, or -10

$x = 1$, or $-2\frac{1}{2}$, *Ans.*

EXAMPLES.

Solve the following equations by completing the square:

3. $2x^2 - 3x = 2$.
4. $3x^2 - 2x = 21$.
5. $4x^2 + x = 18$.
6. $2x^2 - 7x = -6$.
7. $5x^2 - 3x = 68$.
8. $3x^2 - 20x = -25$.
9. $4x^2 - x = 150$.
10. $2x^2 + 7x = 22$.
11. $x^2 + x = 20$.
12. $x^2 + 9x = -20$.
13. $6x^2 + 30x = -36$.
14. $3x^2 + 18x = -15$.
15. $6x^2 - 3x = 45$.
16. $3x^2 + 8x = 28$.

17. $2x^2 + 7x = 9$.

18. $ax^2 + dx = m$.

19. $mx^2 + mx = p$.

20. $x^2 + (n + 1)x = m$.

21. $(x - a)^2 + b = a^2$.

22. $x^2 - (a + b)x = -ab$.

23. $2x^2 + 3x = 13$.

24. $5x^2 + 7x = 9$.

25. $3x^2 + 2x = 15$.

26. $2x^2 + 5x = 11$.

SECOND METHOD OF WRITING THE RESULT BY FORMULA.

223. It was shown in Art. 222 that if $ax^2 + bx = c$, then

$$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}. \qquad (1)$$

Using this result as a *formula*, the roots of a quadratic of the form $ax^2 + bx = c$ can be written without completing the square. Thus,

1. Solve the equation $2x^2 - 3x = 2$.

In this case, $a = 2$, $b = -3$, and $c = 2$; substituting these values in (1),

$$x = \frac{3 \pm \sqrt{16 + 9}}{4} = 2, \text{ or } -\tfrac{1}{2}, \textit{ Ans.}$$

2. Solve the equation $5x^2 + x = 18$.

In this case, $a = 5$, $b = 1$, and $c = 18$; therefore,

$$x = \frac{-1 \pm \sqrt{360 + 1}}{10} = 1\tfrac{4}{5}, \text{ or } -2, \textit{ Ans.}$$

EXAMPLES.

Write the roots to the following equations:

3. $2x^2 - 6x = 20$.

4. $3x^2 - 10x = -3$.

5. $5x^2 - 6x = 8$.

6. $x^2 + 3x = 4$.

7. $x^2 + 3x = 5$.

8. $x^2 - 3x = 5$.

9. $2x^2 - 7x = 4$.

10. $mx^2 + nx = p$.

11. $x^2 - 3(a + b)x = -9ab$.

12. $(x + 3)^2 + 10(x - 1)^2 = 35$.

13. $x^2 - (2x - 3)^2 = -24$.

14. $(x + a)^2 - (x - a)^2 = b$.

15. $(a+b)x^2 + 3(a+b)x = a^2 - b^2.$

16. $x^2 - (m - n + p)x = (n - m)p.$

17. $(x+4)(x-4) - (8x-3) = -25.$

18. $\frac{5}{x} + \frac{x}{5} = 6.$

19. $\frac{x^2-3}{x} + \frac{x^2+3}{2x} = 4.$

20. $x + \frac{b}{cx} = a.$

21. $\frac{2}{x+1} + \frac{3}{x+2} = \frac{4}{x+3}.$

22. $\frac{x}{x-1} - \frac{x}{x-2} = \frac{3}{x-3}.$

23. $\frac{x}{8-x} + \frac{3x}{x-5} = 21.$

24. $5x - \frac{24}{x} = 4x + 2.$

25. $\frac{x}{x-7} + \frac{x+7}{4} = 8\frac{1}{2}.$

26. $\frac{1}{x-1} - \frac{2}{x-2} = \frac{3}{x-3}.$

27. $\frac{x}{x-a} - \frac{x+a}{x} = b.$

28. $\frac{2x+a}{3} + \frac{x^2-a^2}{x} = -\frac{5a}{x}.$

29. $\sqrt{x} + \frac{1}{\sqrt{x}} = 2\frac{1}{2}.$

MISCELLANEOUS EXAMPLES.

224. The results to the following examples may be written by either of the preceding methods. The student should select the method best adapted to each equation.

1. $4x^2 - 3x = x^2 + 60.$

2. $-x^2 + x = -30.$

3. $x - 15x^{-1} = -2.$

4. $x^2 - 4x - 21 = 0.$

5. $ax^2 - a^2x = 1.$

6. $(x+a)^3 - (x-a)^3 = 3a^3.$

7. $(x+3)^3 - (x-3)^3 = 504.$

8. $x^3 - (x+a)^3 = b.$

9. $5x + 2 = 3x^2.$

10. $\sqrt{2x+15} + \sqrt{x-4} = 6.$

11. $(n^2+1)x = nx^2 + n.$

12. $(ax-1)^2 + (x+a)^2 = (a^4-1)x.$

13. $\sqrt{x} + 1 = \frac{x^2}{\sqrt{x} - 1}.$

14. $\sqrt{2x+13}+\sqrt{3x-2}=9.$

15. $\dfrac{x^2-1}{x^2+1}=\dfrac{n+1}{n-1}.$

16. $\dfrac{x}{4-x}+\dfrac{4-x}{x}=3\tfrac{1}{3}.$

17. $\dfrac{x-4}{x}-\dfrac{x}{x+4}=-\dfrac{4}{15}.$

18. $x^2+2px=q.$

19. $x^2-2px=q.$

20. $x^2+2px=-q.$

21. $x^2-2px=-q.$

22. $(x-2)(x-3)=30.$

23. $(x-2)^3-(x-3)^3=19.$

24. $x^2+ax=b.$

25. $x^2-nx-mx=mn.$

26. $x^2-2px=q^3-p^2.$

27. $x^2+2mx=m^2.$

28. $5x^{-1}-2x=-9.$

29. $2x-4=70x^{-1}.$

30. $\dfrac{4}{5-x}+\dfrac{3}{4-x}=\dfrac{2}{3-x}.$

31. $\dfrac{9-x}{9+x}-\dfrac{9+x}{9-x}=-3\tfrac{3}{14}.$

32. $\dfrac{x-2}{x+3}+\dfrac{x-4}{x+5}=\dfrac{19}{40}.$

33. $\dfrac{x}{4}+\dfrac{5}{x}=\dfrac{x}{6}+\dfrac{7}{x}+\dfrac{5}{x}.$

34. $4\left(\dfrac{x-2}{x+2}\right)+\dfrac{x+2}{x-2}=\dfrac{13}{3}.$

35. $\dfrac{x+1}{x-1}=\dfrac{x^2-1}{9}.$

36. $\sqrt{3x+4}+\sqrt{2x+1}=\sqrt{12x+1}.$

37. $x^2-2(m-n+p)x=4(n-m)p.$

38. $x+\sqrt{1+x^2}=\dfrac{3}{\sqrt{1+x^2}}.$

39. $\sqrt{x^2-2x+1}=2(x-4).$

40. $(m+n)^2x^2-(m^2-n^2)x=mn.$

41. $(x-1)(x-2)(x-3)=x^3-58.$

EQUATIONS IN QUADRATIC FORM.

225. An equation is in the **Quadratic Form** when it contains only two powers of an unknown quantity, and the exponent of one power is twice that of the other; as,

$$x^4 + 2x^2 = 5.$$

$$x + 3x^{\frac{1}{2}} = 4.$$

$$(x - 3)^3 + (x - 3)^{\frac{3}{2}} = 5.$$

CASE I.

226. When the unknown term of the quadratic is a monomial.

1. Solve the equation $x^4 + 6x^2 = 40$.

Writing the result by Art. 221,

$$x^2 = -3 \pm \sqrt{40 + 9} = 4, \text{ or } -10,$$

And $x = \pm 2$, or $\pm \sqrt{-10}$, *Ans.*

2. Solve the equation $x^{\frac{2}{3}} + x^{\frac{1}{3}} = 6$.

Writing the result by Art. 223,

$$x^{\frac{1}{3}} = \frac{-1 \pm \sqrt{24 + 1}}{2} = 2, \text{ or } -3,$$

And $x = 8$, or -27, *Ans.*

3. Solve the equation $ax^{2n} + bx^n = c$.

Writing the result by Art. 223,

$$x^n = \frac{-b \pm \sqrt{4ac + b^2}}{2a},$$

And $x = \left(\dfrac{-b \pm \sqrt{4ac + b^2}}{2a}\right)^{\frac{1}{n}}$.

EXAMPLES.

Solve the following equations:

4. $x^4 - 8x^2 = 9$.

5. $x^6 - 5x^3 = -4$.

6. $3\sqrt[3]{x} + 2\sqrt[3]{x^2} = 2$.

7. $x^{2n} - 8x^n = 7$.

8. $x^6 - 7x^3 = 8.$

9. $x + \sqrt{x} = 20.$

10. $2x^4 - 3x^2 = 20.$

11. $2x^{\frac{1}{3}} + 5x^{\frac{2}{3}} = 24.$

12. $3x^{\frac{1}{4}} + 2x^{\frac{1}{2}} = 14.$

13. $2x^3 + 5x^{\frac{3}{2}} = 3.$

14. $ax^n - bx^{\frac{n}{2}} = c.$

15. $mx^{\frac{2}{3}} - mx^{\frac{1}{3}} = n^2.$

16. $x^8 - 9x^4 = 112.$

17. $x^{10} - 31x^5 = 32.$

18. $2x^2 + 3x^{-2} = 18\frac{1}{3}.$

19. $x^{\frac{1}{5}} + \sqrt[5]{x^2} = 12.$

20. $\dfrac{3\sqrt{x} + 7}{\sqrt{x} + 3} = \dfrac{\sqrt{x} + 5}{\sqrt{x}}.$

21. $x - 1 = \dfrac{2\sqrt{x} - 2}{x^{\frac{1}{2}}}.$

22. $x - 4 = \dfrac{3x + 6\sqrt{x}}{x}.$

23. $\dfrac{x - \sqrt{x^2 - 4}}{x + \sqrt{x^2 - 4}} = (x - 4)^2.$

24. $\dfrac{x}{\sqrt{x} + \sqrt{x - 5}} + \dfrac{x}{\sqrt{x} - \sqrt{x - 5}} = \dfrac{10}{\sqrt{x}}.$

25. $\sqrt{3x + 1} + \sqrt{2x - 1} = \sqrt{9x + 4}.$

CASE II.

227. When the unknown term of the quadratic is a polynomial.

In this case the polynomial may be regarded as a single quantity, and solved with reference to that quantity.

1. Solve the equation $(x - 2)^2 + 6(x - 2) = 7.$

Writing the result by ART. 221,

$$x - 2 = -3 \pm \sqrt{7 + 9} = 1, \text{ or } -7,$$

And, $x = 3$, or -5, *Ans.*

2. Solve the equation $(x^2 - 3x)^2 - 4(x^2 - 3x) = 60$.

Writing the result by ART. 221,

$$x^2 - 3x = 2 \pm \sqrt{60 + 4} = 10, \text{ or } -6.$$

Taking $x^2 - 3x = 10$, and writing the result,

$$x = \frac{3 \pm \sqrt{40 + 9}}{2} = 5, \text{ or } -2, \textit{ Ans.}$$

And taking $x^2 - 3x = -6$, and writing the result,

$$x = \frac{3 \pm \sqrt{-24 + 9}}{2} = \frac{3 \pm \sqrt{-15}}{2}, \textit{ Ans.}$$

EXAMPLES.

Solve the following equations:

3. $(x^2 + 4)^2 - 4(x^2 + 4) = 32$.
4. $(x^2 - 16)^2 - 10(x^2 - 16) = 200$.
5. $(x^2 - 3x)^2 - 8(x^2 - 3x) = 20$.
6. $(2x^2 - 3x)^2 - 3(2x^2 - 3x) = -2$.
7. $(x^2 - 6x)^{\frac{1}{2}} + 5(x^2 - 6x)^{\frac{1}{4}} = 14$.
8. $\left(\frac{5}{x} + x\right)^2 - \left(\frac{5}{x} + x\right) = 30$.
9. $\sqrt{\frac{64}{x} + x} + \sqrt[4]{\frac{64}{x} + x} = 6$.
10. $2(3x^2 - 2x - 60)^2 - 4(3x^2 - 2x - 60) = 30$.
11. $(x^2 - 7)^2 + 6x^2 = 58$.
12. $(x^2 + 5)^2 - 7x^2 = 133$.
13. $2(x^2 + x - 5)^2 - 3x^2 - 3x = 390$.
14. $x^2 + 5x - \sqrt{x^2 + 5x + 14} = 42$.
15. $2x^2 + 7x + \sqrt{2x^2 + 3x + 9} = 33 + 4x$.
16. $5(x^2 + 3x) - 3\sqrt{x^2 + 3x - 1} = 41$.
17. $x^4 + 10x^3 + 22x^2 - 15x - 504 = 0$.

SUGGESTION.—$x^4 + 10x^3 + 25x^2 - 3x^2 - 15x = 504$.

Or $(x^2 + 5x)^2 - 3(x^2 + 5x) = 504$.

18. $8x^4 - 24x^3 + 12x^2 + 9x - 135 = 0.$

19. $\left(\frac{x}{x-2}\right)^2 + \left(\frac{x}{x-2}\right) = 6.$

20. $\sqrt{(x-2)^5} + \sqrt{(x-2)^3} = 12\sqrt{(x-2)}.$

21. $3x^4 - 18x^3 + 23x^2 + 12x - 32 = 0.$

22. $(x^2 + 4x - 7)^2 - 3x^2 - 12x = 529.$

23. $(x^2 + x + 2)(x + 1) = 2.$

24. $x^2 + \frac{1}{x^2} = m + \frac{1}{m}.$

PROBLEMS PRODUCING AFFECTED QUADRATICS.

228.—1. The sum of two numbers is 14, and their product is 40; what are the numbers?

Let x = one number.

Then, $14 - x$ = the other number,

By the condition, $x(14 - x) = 40.$

Expanding, $14x - x^2 = 40$

And $x^2 - 14x = -40.$

Writing the result, $x = 7 \pm \sqrt{49 - 40} = 10$, or 4, *Ans.*

And $14 - x = 4$, or 10, *Ans.*

2. Find two numbers whose sum is 24, and whose product is 140.

3. Find two numbers whose difference is 2, and whose product is 24.

4. Divide 32 into two such parts that their product will be 240.

5. Find a number which, being increased by six times its square root, equals 27.

6. The sum of two numbers is 16, and the sum of their squares is 160; find the numbers.

7. The difference of two numbers is 6, and the sum of their *squares is 90; find the* numbers.

$$4a^2x^2 + 4abx = 4ac \tag{1}$$

Completing the square by Art. 220,

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2 \tag{2}$$

Extracting the square root, $2ax + b = \pm\sqrt{4ac + b^2}$ (3)

Transposing, $2ax = -b \pm \sqrt{4ac + b^2}$

And, $x = \dfrac{-b \pm \sqrt{4ac + b^2}}{2a}$, *Ans.*

Therefore, to solve an affected quadratic equation without using fractions,

RULE.

Reduce the equation to the form $ax^2 + bx = c$.

Multiply the equation by 4 *times the coefficient of* x^2, *and add the square of the coefficient of* x *to both members.*

Extract the square root, and find the value of x *in the resulting equation.*

2. Solve the equation $2x^2 + 3x = 5$.

Multiplying both members by 4 times 2, or 8,

$$16x^2 + 24x = 40,$$

Adding to each member the square of 3,

$$16x^2 + 24x + 9 = 49,$$

Extracting the square root, $4x + 3 = \pm 7$,

Transposing, $4x = -3 \pm 7 = 4$, or -10

$x = 1$, or $-2\frac{1}{2}$, *Ans.*

EXAMPLES.

Solve the following equations by completing the square:

3. $2x^2 - 3x = 2$.
4. $3x^2 - 2x = 21$.
5. $4x^2 + x = 18$.
6. $2x^2 - 7x = -6$.
7. $5x^2 - 3x = 68$.
8. $3x^2 - 20x = -25$.
9. $4x^2 - x = 150$.
10. $2x^2 + 7x = 22$.
11. $x^2 + x = 20$.
12. $x^2 + 9x = -20$.
13. $6x^2 + 30x = -36$.
14. $3x^2 + 18x = -15$.
15. $6x^2 - 3x = 45$.
16. $3x^2 + 8x = 28$.

17. $2x^2 + 7x = 9$.
18. $ax^2 + dx = m$.
19. $mx^2 + mx = p$.
20. $x^2 + (n + 1)x = m$.
21. $(x - a)^2 + b = a^2$.
22. $x^2 - (a + b)x = -ab$.
23. $2x^2 + 3x = 13$.
24. $5x^2 + 7x = 9$.
25. $3x^2 + 2x = 15$.
26. $2x^2 + 5x = 11$.

SECOND METHOD OF WRITING THE RESULT BY FORMULA.

223. It was shown in ART. 222 that if $ax^2 + bx = c$, then

$$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}. \qquad (1)$$

Using this result as a *formula*, the roots of a quadratic of the form $ax^2 + bx = c$ can be written without completing the square. Thus,

1. Solve the equation $2x^2 - 3x = 2$.

In this case, $a = 2$, $b = -3$, and $c = 2$; substituting these values in (1),

$$x = \frac{3 \pm \sqrt{16 + 9}}{4} = 2, \text{ or } -\tfrac{1}{2}, \textit{ Ans.}$$

2. Solve the equation $5x^2 + x = 18$.

In this case, $a = 5$, $b = 1$, and $c = 18$; therefore,

$$x = \frac{-1 \pm \sqrt{360 + 1}}{10} = 1\tfrac{4}{5}, \text{ or } -2, \textit{ Ans.}$$

EXAMPLES.

Write the roots to the following equations:

3. $2x^2 - 6x = 20$.
4. $3x^2 - 10x = -3$.
5. $5x^2 - 6x = 8$.
6. $x^2 + 3x = 4$.
7. $x^2 + 3x = 5$.
8. $x^2 - 3x = 5$.
9. $2x^2 - 7x = 4$.
10. $mx^2 + nx = p$.
11. $x^2 - 3(a + b)x = -9ab$.
12. $(x + 3)^2 + 10(x - 1)^2 = 35$.
13. $x^2 - (2x - 3)^2 = -24$.
14. $(x + a)^2 - (x - a)^2 = b$.

15. $(a+b)x^2 + 3(a+b)x = a^2 - b^2.$

16. $x^2 - (m-n+p)x = (n-m)p.$

17. $(x+4)(x-4) - (8x-3) = -25.$

18. $\frac{5}{x} + \frac{x}{5} = 6.$

19. $\frac{x^2-3}{x} + \frac{x^2+3}{2x} = 4.$

20. $x + \frac{b}{cx} = a.$

21. $\frac{2}{x+1} + \frac{3}{x+2} = \frac{4}{x+3}.$

22. $\frac{x}{x-1} - \frac{x}{x-2} = \frac{3}{x-3}.$

23. $\frac{x}{8-x} + \frac{3x}{x-5} = 21.$

24. $5x - \frac{24}{x} = 4x + 2.$

25. $\frac{x}{x-7} + \frac{x+7}{4} = 8\frac{1}{2}.$

26. $\frac{1}{x-1} - \frac{2}{x-2} = \frac{3}{x-3}.$

27. $\frac{x}{x-a} - \frac{x+a}{x} = b.$

28. $\frac{2x+a}{3} + \frac{x^2-a^2}{x} = -\frac{5a}{x}.$

29. $\sqrt{x} + \frac{1}{\sqrt{x}} = 2\frac{1}{2}.$

MISCELLANEOUS EXAMPLES.

224. The results to the following examples may be written by either of the preceding methods. The student should select the method best adapted to each equation.

1. $4x^2 - 3x = x^2 + 60.$

2. $-x^2 + x = -30.$

3. $x - 15x^{-1} = -2.$

4. $x^2 - 4x - 21 = 0.$

5. $ax^2 - a^2x = 1.$

6. $(x+a)^3 - (x-a)^3 = 3a^3.$

7. $(x+3)^3 - (x-3)^3 = 504.$

8. $x^3 - (x+a)^3 = b.$

9. $5x + 2 = 3x^2.$

10. $\sqrt{2x+15} + \sqrt{x-4} = 6.$

11. $(n^2+1)x = nx^2 + n.$

12. $(ax-1)^2 + (x+a)^2 = (a^4-1)x.$

13. $\sqrt{x} + 1 = \frac{x^2}{\sqrt{x}-1}.$

14. $\sqrt{2x+13}+\sqrt{3x-2}=9$.

15. $\dfrac{x^2-1}{x^2+1}=\dfrac{n+1}{n-1}$.

16. $\dfrac{x}{4-x}+\dfrac{4-x}{x}=3\tfrac{1}{3}$.

17. $\dfrac{x-4}{x}-\dfrac{x}{x+4}=-\dfrac{4}{15}$.

18. $x^2+2px=q$.

19. $x^2-2px=q$.

20. $x^2+2px=-q$.

21. $x^2-2px=-q$.

22. $(x-2)(x-3)=30$.

23. $(x-2)^3-(x-3)^3=19$.

24. $x^2+ax=b$.

25. $x^2-nx-mx=mn$.

26. $x^2-2px=q^3-p^2$.

27. $x^2+2mx=m^2$.

28. $5x^{-1}-2x=-9$.

29. $2x-4=70x^{-1}$.

30. $\dfrac{4}{5-x}+\dfrac{3}{4-x}=\dfrac{2}{3-x}$.

31. $\dfrac{9-x}{9+x}-\dfrac{9+x}{9-x}=-3\tfrac{3}{14}$.

32. $\dfrac{x-2}{x+3}+\dfrac{x-4}{x+5}=\dfrac{19}{40}$.

33. $\dfrac{x}{4}+\dfrac{5}{x}=\dfrac{x}{6}+\dfrac{7}{x}+\dfrac{5}{x}$.

34. $4\left(\dfrac{x-2}{x+2}\right)+\dfrac{x+2}{x-2}=\dfrac{13}{3}$.

35. $\dfrac{x+1}{x-1}=\dfrac{x^2-1}{9}$.

36. $\sqrt{3x+4}+\sqrt{2x+1}=\sqrt{12x+1}$.

37. $x^2-2(m-n+p)x=4(n-m)p$.

38. $x+\sqrt{1+x^2}=\dfrac{3}{\sqrt{1+x^2}}$.

39. $\sqrt{x^2-2x+1}=2(x-4)$.

40. $(m+n)^2x^2-(m^2-n^2)x=mn$.

41. $(x-1)(x-2)(x-3)=x^3-58$.

EQUATIONS IN QUADRATIC FORM.

225. An equation is in the **Quadratic Form** when it contains only two powers of an unknown quantity, and the exponent of one power is twice that of the other; as,

$$x^4 + 2x^2 = 5.$$
$$x + 3x^{\frac{1}{2}} = 4.$$
$$(x - 3)^3 + (x - 3)^{\frac{3}{2}} = 5.$$

CASE I.

226. When the unknown term of the quadratic is a monomial.

1. Solve the equation $x^4 + 6x^2 = 40$.

Writing the result by Art. 221,

$$x^2 = -3 \pm \sqrt{40 + 9} = 4, \text{ or } -10,$$

And $\quad x = \pm 2, \text{ or } \pm\sqrt{-10}$, *Ans.*

2. Solve the equation $x^{\frac{2}{3}} + x^{\frac{1}{3}} = 6$.

Writing the result by Art. 223,

$$x^{\frac{1}{3}} = \frac{-1 \pm \sqrt{24 + 1}}{2} = 2, \text{ or } -3,$$

And $\quad x = 8, \text{ or } -27$, *Ans.*

3. Solve the equation $ax^{2n} + bx^n = c$.

Writing the result by Art. 223,

$$x^n = \frac{-b \pm \sqrt{4ac + b^2}}{2a},$$

And
$$x = \left(\frac{-b \pm \sqrt{4ac + b^2}}{2a}\right)^{\frac{1}{n}}.$$

EXAMPLES.

Solve the following equations:

4. $x^4 - 8x^2 = 9$.

5. $x^6 - 5x^3 = -4$.

6. $3\sqrt[3]{x} + 2\sqrt[3]{x^2} = 2$.

7. $x^{2n} - 8x^n = 7$.

8. $x^6 - 7x^3 = 8.$

9. $x + \sqrt{x} = 20.$

10. $2x^4 - 3x^2 = 20.$

11. $2x^{\frac{1}{3}} + 5x^{\frac{2}{3}} = 24.$

12. $3x^{\frac{1}{4}} + 2x^{\frac{1}{2}} = 14.$

13. $2x^3 + 5x^{\frac{3}{2}} = 3.$

14. $ax^n - bx^{\frac{n}{2}} = c.$

15. $mx^{\frac{2}{3}} - mx^{\frac{1}{3}} = n^2.$

16. $x^8 - 9x^4 = 112.$

17. $x^{10} - 31x^5 = 32.$

18. $2x^2 + 3x^{-2} = 18\frac{1}{3}.$

19. $x^{\frac{1}{5}} + \sqrt[5]{x^2} = 12.$

20. $\dfrac{3\sqrt{x} + 7}{\sqrt{x} + 3} = \dfrac{\sqrt{x} + 5}{\sqrt{x}}.$

21. $x - 1 = \dfrac{2\sqrt{x} - 2}{x^{\frac{1}{2}}}.$

22. $x - 4 = \dfrac{3x + 6\sqrt{x}}{x}.$

23. $\dfrac{x - \sqrt{x^2 - 4}}{x + \sqrt{x^2 - 4}} = (x - 4)^2.$

24. $\dfrac{x}{\sqrt{x} + \sqrt{x - 5}} + \dfrac{x}{\sqrt{x} - \sqrt{x - 5}} = \dfrac{10}{\sqrt{x}}.$

25. $\sqrt{3x + 1} + \sqrt{2x - 1} = \sqrt{9x + 4}.$

CASE II.

227. When the unknown term of the quadratic is a polynomial.

In this case the polynomial may be regarded as a single quantity, and solved with reference to that quantity.

1. Solve the equation $(x - 2)^2 + 6(x - 2) = 7.$

Writing the result by ART. 221,

$$x - 2 = -3 \pm \sqrt{7 + 9} = 1, \text{ or } -7,$$

And, $x = 3$, or -5, *Ans.*

2. Solve the equation $(x^2 - 3x)^2 - 4(x^2 - 3x) = 60$.

Writing the result by ART. 221,

$$x^2 - 3x = 2 \pm \sqrt{60 + 4} = 10, \text{ or } -6.$$

Taking $x^2 - 3x = 10$, and writing the result,

$$x = \frac{3 \pm \sqrt{40 + 9}}{2} = 5, \text{ or } -2, \textit{ Ans.}$$

And taking $x^2 - 3x = -6$, and writing the result,

$$x = \frac{3 \pm \sqrt{-24 + 9}}{2} = \frac{3 \pm \sqrt{-15}}{2}, \textit{ Ans.}$$

EXAMPLES.

Solve the following equations:

3. $(x^2 + 4)^2 - 4(x^2 + 4) = 32.$
4. $(x^2 - 16)^2 - 10(x^2 - 16) = 200.$
5. $(x^2 - 3x)^2 - 8(x^2 - 3x) = 20.$
6. $(2x^2 - 3x)^2 - 3(2x^2 - 3x) = -2.$
7. $(x^2 - 6x)^{\frac{1}{2}} + 5(x^2 - 6x)^{\frac{1}{4}} = 14.$
8. $\left(\frac{5}{x} + x\right)^2 - \left(\frac{5}{x} + x\right) = 30.$
9. $\sqrt{\frac{64}{x} + x} + \sqrt[4]{\frac{64}{x} + x} = 6.$
10. $2(3x^2 - 2x - 60)^2 - 4(3x^2 - 2x - 60) = 30.$
11. $(x^2 - 7)^2 + 6x^2 = 58.$
12. $(x^2 + 5)^2 - 7x^2 = 133.$
13. $2(x^2 + x - 5)^2 - 3x^2 - 3x = 390.$
14. $x^2 + 5x - \sqrt{x^2 + 5x + 14} = 42.$
15. $2x^2 + 7x + \sqrt{2x^2 + 3x + 9} = 33 + 4x.$
16. $5(x^2 + 3x) - 3\sqrt{x^2 + 3x - 1} = 41.$
17. $x^4 + 10x^3 + 22x^2 - 15x - 504 = 0.$

SUGGESTION.—$x^4 + 10x^3 + 25x^2 - 3x^2 - 15x = 504.$

Or $(x^2 + 5x)^2 - 3(x^2 + 5x) = 504.$

18. $8x^4 - 24x^3 + 12x^2 + 9x - 135 = 0.$

19. $\left(\dfrac{x}{x-2}\right)^2 + \left(\dfrac{x}{x-2}\right) = 6.$

20. $\sqrt{(x-2)^5} + \sqrt{(x-2)^3} = 12\sqrt{(x-2)}.$

21. $3x^4 - 18x^3 + 23x^2 + 12x - 32 = 0.$

22. $(x^2 + 4x - 7)^2 - 3x^2 - 12x = 529.$

23. $(x^2 + x + 2)(x + 1) = 2.$

24. $x^2 + \dfrac{1}{x^2} = m + \dfrac{1}{m}.$

PROBLEMS PRODUCING AFFECTED QUADRATICS.

228.—1. The sum of two numbers is 14, and their product is 40; what are the numbers?

Let x = one number.
Then, $14 - x$ = the other number,
By the condition, $x(14 - x) = 40.$
Expanding, $14x - x^2 = 40$
And $x^2 - 14x = -40.$
Writing the result, $x = 7 \pm \sqrt{49 - 40} = 10$, or 4, *Ans.*
And $14 - x = 4$, or 10, *Ans.*

2. Find two numbers whose sum is 24, and whose product is 140.

3. Find two numbers whose difference is 2, and whose product is 24.

4. Divide 32 into two such parts that their product will be 240.

5. Find a number which, being increased by six times its square root, equals 27.

6. The sum of two numbers is 16, and the sum of their squares is 160; find the numbers.

7. The difference of two numbers is 6, and the sum of their *squares is 90; find* the numbers.

8. A man sold a cow for $24, and gained as many per cent. as the cow cost him dollars; find the cost of the cow.

9. A distributed $120 among a number of persons: if each person had received $2 more, he would have received as many dollars as there were persons; how many persons were there?

10. A number of persons hired a pleasure-boat for $60: if there had been 5 more persons, the number of dollars each would have then paid would have been 6 less than the number of persons; how many persons were there?

11. If 10 be subtracted from a number, the difference between the remainder and its square will be 30; find the number.

12. A flower bed is 8 feet wide and 10 feet long; how wide must a walk around the outside of it be to contain as many square feet as the bed?

13. A yard is 10 feet wide and 12 feet long: a walk is made halfway around, which contains $\frac{1}{8}$ of the entire area; how wide is the walk?

14. A man sold his horse for $144, and gained as many per cent. as the horse cost him dollars; what was the cost of the horse?

15. The number of square inches in the surface of a cubical block exceeds the number of linear inches in the sum of its edges by 288; what is its volume?

16. Three times the square of a number increased by 5, exceeds five times the number by 177; what is the number required?

17. A traveled 150 miles: if he had gone 5 miles more an hour, he would have performed the journey in 5 hours less time. Required the rate per hour.

18. If a bar of iron weighing 60 pounds be drawn out 3 feet longer, it will weigh one pound less per linear foot. *How long is it?*

19. A and B have a distance of 100 miles to travel: A, who travels a mile more an hour than B, starts 2 hours later and arrives 3 hours earlier than B. Required the rate of each per hour.

20. A set out from M to N; B at the same time from N to M. A arrived at N 3 hours and B arrived at M 12 hours after they had met. In what time did each complete the journey?

21. A ladder whose foot is in the street reaches a window on one side 40 feet high, and on the other side a window 30 feet high. What is the width of the street, and what the length of the ladder, if the two positions of the ladder are at right angles with each other?

22. The hypotenuse of a right triangle is 100 feet, and the area of the triangle is 2,400 square feet; find the length of the other two sides.

23. A sold goods for $24, and lost as many per cent. as the goods cost him dollars. Find the cost of the goods.

24. A rectangular farm is three times as long as it is wide. If it is made 20 rods wider and 40 rods longer, its area will be doubled. Find the area of the farm.

25. There is a number consisting of two digits, whose sum is 12 and the sum of whose squares is 74. Find the number.

26. M and N start at the same time to travel 120 miles; M travels 2 miles an hour faster than N, and arrives two hours earlier; at what rate per hour did each travel?

27. The product of four consecutive numbers is 360. Find the numbers.

28. What will be the price of eggs per dozen, when 3 less for 12 cents raises the price 8 cents per dozen?

29. What number is that whose cube added to its square equals 16 times the next higher number?

30. What number is that which if diminished by 7 will equal 6 divided by the square root of the number?

31. How much are eggs per dozen if 6 dozen cost as many *cents as the number of* eggs you can buy for 8 cents?

QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

229. The **Degree** of an equation containing two or more unknown quantities is determined by the greatest sum of the exponents of the unknown quantities in any term.

Thus, $xy = 4$, is an equation of the 2d degree.
$x^3 + xy = 6$, is an equation of the 3d degree.

230. A **Homogeneous** equation is one in which the sum of the exponents of the unknown quantities in each term is the same; as,

$$xy + y^2 = 20$$
$$x^2 + y^2 = 24$$

231. **Quadratic Equations** containing two unknown quantities can be solved by the ordinary rules for quadratics, if they fall under one of the following cases:

CASE I.

232. When one equation is simple and the other quadratic.

Equations of this case can always be solved by finding the value of one of the unknown quantities from the simple equation, and then substituting the result in the other equation.

1. Solve the equations $\begin{cases} 3x + y = 9. & (1) \\ x^2 + y^2 = 13. & (2) \end{cases}$

From (1), $y = 9 - 3x$ (3)
Substituting in (2), $x^2 + 81 - 54x + 9x^2 = 13$ (4)
Collecting, $10x^2 - 54x = - 68$ (5)
Or $5x^2 - 27x = - 34$ (6)
Writing the result, $x = \dfrac{27 \pm \sqrt{- 680 + 729}}{10}$

$x = 2$, or $3\frac{2}{5}$, Ans.

EXAMPLES.

Solve the following equations:

2. $\begin{cases} 2x + y = 9. \\ 3x^2 + 2y^2 = 62. \end{cases}$

3. $\begin{cases} x + 4y = 8. \\ 3x^2 + 2y^2 = 50. \end{cases}$

4. $\begin{cases} 2x^2 + xy = 24. \\ x + 2y = 7. \end{cases}$

5. $\begin{cases} 4x^2 + 3y^2 = 511. \\ 3x + 2y = 27. \end{cases}$

6. $\begin{cases} x + y = 4. \\ x^2 + 10y = 19. \end{cases}$

7. $\begin{cases} 2x^2 - 3xy = 35. \\ 3x + 2y = 27. \end{cases}$

An excellent method, when possible, is to find the value of $x + y$ and $x - y$, or a similar expression, and then find the value of x by addition and the value of y by subtraction. Thus,

8. Solve the equations $\begin{cases} x + y = 12. & (1) \\ x^2 - y^2 = 24. & (2) \end{cases}$

Dividing (2) by (1), $x - y = 2$ (3)

Adding (1) and (3), $2x = 14$

$x = 7$, *Ans.*

Subtracting (3) from (1), $2y = 10$

$y = 5$, *Ans.*

9. Solve the equations $\begin{cases} x + y = 11. & (1) \\ xy = 24. & (2) \end{cases}$

Squaring (1), $x^2 + 2xy + y^2 = 121$ (3)

Subtracting 4 times (2), $4xy = 96$ (4)

$x^2 - 2xy + y^2 = 25$ (5)

Extracting the square root, $x - y = \pm 5$ (6)

Adding (1) and (6) $x + y = 11$

$2x = 16, \text{ or } 6$

$x = 8, \text{ or } 3$, *Ans.*

Subtracting (6) from (1), $2y = 6, \text{ or } 16$

$y = 3, \text{ or } 8$, Ans.

10. $\begin{cases} x - y = 2. \\ x^2 - y^2 = 16. \end{cases}$

11. $\begin{cases} x + y = 6. \\ x^2 - y^2 = 12. \end{cases}$

12. $\begin{cases} x + y = 7. \\ xy = 12. \end{cases}$

13. $\begin{cases} x - y = 2. \\ xy = 8. \end{cases}$

14. $\begin{cases} x + y = 9. \\ x^2 + y^2 = 45. \end{cases}$

15. $\begin{cases} x - y = 1. \\ x^2 + y^2 = 25. \end{cases}$

16. $\begin{cases} 2x - y = 10. \\ xy = 12. \end{cases}$

17 $\begin{cases} 8x^3 - y^3 = 37. \\ 2x - y = 1. \end{cases}$

18. $\begin{cases} x^2 + 5xy + y^2 = 79. \\ x + y = 7. \end{cases}$

19. $\begin{cases} \dfrac{x}{4} + \dfrac{y}{5} = 4. \\ \dfrac{x^2}{16} + \dfrac{y^2}{25} = 8. \end{cases}$

20. $\begin{cases} x + y = 9. \\ x^3 + y^3 = 351. \end{cases}$

21. $\begin{cases} x - y = 2. \\ x^3 - y^3 = 98. \end{cases}$

22. $\begin{cases} x - y = 1. \\ x^3 - y^3 = 61(x - y). \end{cases}$

23. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5. \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = 13. \end{cases}$

24. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = 4. \\ \dfrac{1}{x^2} - \dfrac{1}{y^2} = 24. \end{cases}$

25. $\begin{cases} x - 2y = 1. \\ x^3 - 8y^3 = 61. \end{cases}$

CASE II.

233. When the equations are homogeneous and quadratic.

Equations in this case are most conveniently solved by *letting* $y = vx$.

1. Solve the equations $\begin{cases} x^2 + xy = 12. & (1) \\ xy - y^2 = 2. & (2) \end{cases}$

Let $y = vx$

Substituting in (1), $x^2 + vx^2 = 12$ (3)

Substituting in (2), $vx^2 - v^2x^2 = 2$ (4)

From (3), $x^2 = \dfrac{12}{1+v}$ (5)

From (4), $x^2 = \dfrac{2}{v - v^2}$ (6)

Equating, or comparing, (5) and (6), $\dfrac{12}{1+v} = \dfrac{2}{v - v^2}$ (7)

Dividing by 2 and clearing of fractions,

$$6v - 6v^2 = 1 + v \qquad (8)$$

Solving this equation, $v = \frac{1}{2}$, or $\frac{1}{3}$ (9)

Substituting these values in (5), $x^2 = \dfrac{12}{1+\frac{1}{2}}$ or $\dfrac{12}{1+\frac{1}{3}}$

$$x^2 = 8 \text{ or } 9$$

$$x = \pm 2\sqrt{2} \text{ or } \pm 3, \textit{ Ans.}$$

Since $y = vx$, $y = \frac{1}{2}$ of $\pm 2\sqrt{2} = \pm\sqrt{2}$, *Ans.*

Or $y = \frac{1}{3}$ of $\pm 3 = \pm 1$, *Ans.*

EXAMPLES.

2. $\begin{cases} x^2 - 3xy = -20. \\ x^2 - y^2 = 16. \end{cases}$

3. $\begin{cases} x^2 - xy = 48. \\ xy + y^2 = 20. \end{cases}$

4. $\begin{cases} 2x^2 + 3xy = 126. \\ 3y^2 - 2xy = -9. \end{cases}$

5. $\begin{cases} x^2 + 3y^2 = 52. \\ 2x^2 + 5y^2 = 19. \end{cases}$

6. $\begin{cases} x^2 - xy = 5. \\ x^2 + xy + 2y^2 = 77. \end{cases}$

7. $\begin{cases} x^2 + 2xy - 3y^2 = 5. \\ 2x^2 + 5y^2 = 13. \end{cases}$

8. $\begin{cases} x^2 + 2xy - 3y^2 = 9. \\ 3x^2 - 3xy + y^2 = 13. \end{cases}$

9. $\begin{cases} 3xy - x^2 = 20. \\ 5x^2 - 3xy + y^2 = 89. \end{cases}$

10. $\begin{cases} x^2 - y^2 = 3. \\ x^2 - 2xy + 2y^2 = 2. \end{cases}$

11. $\begin{cases} x^2 + 2xy - 3y^2 = 12. \\ 3x^2 - 5xy + y^2 = 13. \end{cases}$

12. $\begin{cases} x^2 + xy - y^2 = 11. \\ 2x^2 - xy + y^2 = 16. \end{cases}$

13. $\begin{cases} x^2 + xy - y^2 = 11. \\ 5x^2 - 7y^2 = 38. \end{cases}$

CASE III.

234. When the equations are symmetrical with respect to x and y.

An equation is *symmetrical* when the unknown quantities are similarly involved, or when the unknown quantities can change places without destroying the equation.

$$x^2+y^2=25,\ x^2-xy+y^2=12, \text{ and } x^2+y^2+x+y=4$$

are symmetrical equations.

But $x-2y=3$ is not symmetrical; for on interchanging x and y it becomes $2x-y=3$, which is a different equation.

There is no general method for the solution of equations in this case, but the aim should be to find the values of the sum and the difference of the unknown quantities (ART. 232, Ex. 8–25).

Frequently examples of this case can be solved by substituting for the unknown quantities the sum and the difference of two other unknown quantities.

1. Solve the equations $\begin{cases} x+y=6. & (1) \\ x^4+y^4=272. & (2) \end{cases}$

Let $x=u+v$, and $y=u-v$.

Then, from (1), $(u+v)+(u-v)=6$ (3)

And from (2), $(u+v)^4+(u-v)^4=272$ (4)

Collecting (3), $2u=6$

$u=3$ (5)

Expanding (4), $2u^4+12u^2v^2+2v^4=272$ (6)

Or $u^4+6u^2v^2+v^4=136$ (7)

Substituting the value of u from (5) in (7),

$v^4+54v^2=55$ (8)

Writing the result, $v^2=-27\pm\sqrt{55+27^2}$

$v^2=1$

And $v=\pm 1$

Therefore, $x=u+v=3\pm 1=4$ or 2, *Ans.*

$y=u-v=3\mp 1=2$ or 4, *Ans.*

2. Solve the equations $\begin{cases} x^3 - y^3 = 117. & (1) \\ x^2 + xy + y^2 = 39. & (2) \end{cases}$

Dividing (1) by (2), $x - y = 3$ (3)

Squaring, $x^2 - 2xy + y^2 = 9$ (4)

Subtracting (4) from (2), $3xy = 30$

Or $xy = 10$ (5)

Adding (2) and (5), $x^2 + 2xy + y^2 = 49$

Whence, $x + y = \pm 7$ (6)

Adding (3) and (6), $2x = 3 \pm 7 = 10$, or -4

$x = 5$, or -2, *Ans.*

Subtracting (3) from (6), $2y = -3 \pm 7 = 4$, or -10

$y = 2$, or -5, *Ans.*

3. Solve the equations $\begin{cases} x^3 + y^3 = 126. & (1) \\ x^2y + xy^2 = 30. & (2) \end{cases}$

Multiplying (2) by 3, and adding the result to (1),

$x^3 + 3x^2y + 3xy^2 + y^3 = 216$ (3)

Extracting the cube root of (3), $x + y = 6$ (4)

Dividing (2) by (4), $xy = 5$ (5)

Squaring (4), $x^2 + 2xy + y^2 = 36$ (6)

Subtracting 4 times (5) from (6),

$x^2 - 2xy + y^2 = 16$ (7)

Extracting the square root, $x - y = \pm 4$ (8)

Adding (4) and (8), $2x = 10$, or 6

$x = 5$, or 3, *Ans.*

Subtracting (4) from (8), $2y = 6$, or 10

$y = 3$, or 5, *Ans.*

EXAMPLES.

Solve the following equations :

4. $\begin{cases} xy = 18. \\ x^2 + y^2 = 45. \end{cases}$

6. $\begin{cases} x + y = 8. \\ x^3 + y^3 = 224. \end{cases}$

5. $\begin{cases} \sqrt{xy} = 3. \\ \sqrt{x} + \sqrt{y} = 4. \end{cases}$

7. $\begin{cases} xy = 6. \\ x^4 + y^4 = 97. \end{cases}$

8. $\begin{cases} x^2 + xy + y^2 = 13. \\ x^4 + x^2y^2 + y^4 = 91. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 104. \\ x^4 + y^4 = 10{,}016. \end{cases}$

10. $\begin{cases} x^3 + y^3 = 9. \\ x^2y + xy^2 = 6. \end{cases}$

11. $\begin{cases} xy = 8. \\ (x+y)^2 + 3(x+y) = 54. \end{cases}$

12. $\begin{cases} x^3 + y^3 = 35. \\ x^2 - xy + y^2 = 7. \end{cases}$

13. $\begin{cases} x^3 - y^3 = 56. \\ x - y = 2. \end{cases}$

14. $\begin{cases} x + y = 5. \\ x^4 + y^4 = 97. \end{cases}$

15. $\begin{cases} x + y = 3. \\ x^5 + y^5 = 33. \end{cases}$

16. $\begin{cases} x^2 + y^2 = 29. \\ x + y + xy = 17. \end{cases}$

17. $\begin{cases} x^2 + y^2 - 2xy = 4. \\ x^4 + y^4 = 706. \end{cases}$

18. $\begin{cases} x^2 + y^2 + xy = 28. \\ x^4 + y^4 + x^2y^2 = 336. \end{cases}$

19. $\begin{cases} x^3 + y^3 = 344. \\ x + y = 8. \end{cases}$

MISCELLANEOUS EXAMPLES.

235. Many examples, which cannot be classified under any of the cases just considered, can be solved by **special artifices.** The student can become familiar with these only by practice.

1. Solve the equations $\begin{cases} x^3 + y^3 = 9xy. & (1) \\ x + y = 6. & (2) \end{cases}$

Dividing (1) by (2), $x^2 - xy + y^2 = \frac{3}{2}xy$ (3)

$$\therefore x^2 - \tfrac{5}{2}xy + y^2 = 0 \qquad (4)$$

Subtracting (4) from the square of (2),

$$\tfrac{9}{2}xy = 36$$

$$\therefore \tfrac{1}{2}xy = 4 \qquad (5)$$

Adding (4) and (5), and extracting the square root,

$$x - y = \pm 2$$

$$\underline{x + y = 6}$$

$$2x = 8, \text{ or } 4$$

$$x = 4, \text{ or } 2, \textit{ Ans.}$$

$$2y = 4, \text{ or } 8$$

$$y = 2, \text{ or } 4, \textit{ Ans.}$$

2. Solve the equations $\begin{cases} x^2 + y^2 = 13. & (1) \\ x + xy + y = 11. & (2) \end{cases}$

Multiplying (2) by 2, $2x + 2xy + 2y = 22$ (3)

Adding (1) and (3),

$$x^2 + 2xy + y^2 + 2x + 2y = 35$$

Or $(x + y)^2 + 2(x + y) = 35$ (4)

Solving (4) for $x + y$,

$$x + y = -1 \pm 6 = 5, \text{ or } -7 \tag{5}$$

Squaring (5), $x^2 + 2xy + y^2 = 25, \text{ or } 49$ (6)

Subtracting (1) from (6), $2xy = 12, \text{ or } 36$ (7)

Subtracting (7) from (1),

$$x^2 - 2xy + y^2 = 1, \text{ or } -23 \tag{8}$$

Whence, $x - y = \pm 1, \text{ or } \pm \sqrt{-23}$ (9)

Adding (5) and (9), $2x = 5 \pm 1, \text{ or } -7 \pm \sqrt{-23}.$

$$x = 3, 2, \text{ or } \tfrac{1}{2}(-7 \pm \sqrt{-23}), \textit{ Ans.}$$

Subtracting (9) from (5), $2y = 5 \mp 1, \text{ or } -7 \mp \sqrt{-23})$

$$y = 2, 3, \text{ or } \tfrac{1}{2}(-7 \mp \sqrt{-23}), \textit{ Ans.}$$

3. Solve the equations $\begin{cases} x - \sqrt{xy} + y = 7. & (1) \\ x^2 + xy + y^2 = 91. & (2) \end{cases}$

Dividing (2) by (1), $x + \sqrt{xy} + y = 13$ (3)

Subtracting (1) from (3), $2\sqrt{xy} = 6$

$$\sqrt{xy} = 3 \tag{4}$$

Adding (4) to (1), $x + y = 10$ (5)

Squaring (4), $xy = 9$ (6)

Subtracting 3 times (6) from (2), $x^2 - 2xy + y^2 = 64$

Whence, $x - y = \pm 8$ (7)

Adding (5) and (7), $2x = 18, \text{ or } 2$

$$x = 9, \text{ or } 1, \textit{ Ans.}$$

Subtracting (7) from (5), $2y = 2, \text{ or } 18$

$$y = 1, \text{ or } 9, \text{ Ans.}$$

4. Solve the equations $\begin{cases} x^2y + xy^2 = 30. & (1) \\ x^4y^2 + x^2y^4 = 468. & (2) \end{cases}$

Subtracting (2) from the square of (1),

$$2x^3y^3 = 432$$

Whence, $xy = 6$ (3)

Dividing (1) by (3), $x + y = 5$ (4)

Dividing (2) by the square of (3), $x^2 + y^2 = 13$ (5)

Subtracting 2 times (3) from (5),

$$x^2 - 2xy + y^2 = 1$$

Whence, $x - y = \pm 1$ (6)

Adding (4) and (6), $2x = 6$, or 4

$x = 3$, or 2, *Ans.*

Subtracting (6) from (4), $2y = 4$, or 6

$y = 2$, or 3, *Ans.*

EXAMPLES.

Solve the following equations:

5. $\begin{cases} x^2 - y^2 = 16. \\ xy + y^2 = 24. \end{cases}$

6. $\begin{cases} x^3 - y^3 = 98. \\ x - y = 2. \end{cases}$

7. $\begin{cases} x - y = 8. \\ x^2 - y^2 = 80. \end{cases}$

8. $\begin{cases} x^2 + y^2 + 2x = 19. \\ xy + y = 8. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 10. \\ xy - x - y = -1. \end{cases}$

10. $\begin{cases} (x + y)^2 + 4(x + y) = 117. \\ x - y = 7. \end{cases}$

11. $\begin{cases} \sqrt[3]{x} - \sqrt[3]{y} = 1. \\ x - y = 7. \end{cases}$

12. $\begin{cases} x^2 + xy + y^2 = 39. \\ x^4 + x^2y^2 + y^4 = 741. \end{cases}$

13. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = -\dfrac{1}{6}. \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{13}{36}. \end{cases}$

14. $\begin{cases} x^2 + y^2 + x + y = 62. \\ xy = 14. \end{cases}$

15. $\begin{cases} x + y = 3. \\ x^4 + y^4 = 17. \end{cases}$

16. $\begin{cases} x^2 + y^2 = 37. \\ xy + y^2 = 7. \end{cases}$

17. $$\begin{cases} (x+y)(x^2+y^2)=65. \\ x^2y+xy^2=30. \end{cases}$$

18. $$\begin{cases} x^3+y^3=133. \\ x+y=\dfrac{70}{xy}. \end{cases}$$

19. $$\begin{cases} 3x^2y^2+2xy=85. \\ x-y=4. \end{cases}$$

20. $$\begin{cases} x+\sqrt{xy}+y=7. \\ x^2+xy+y^2=21. \end{cases}$$

21. $$\begin{cases} x^4+x^2y^2+y^4=651. \\ x^2+xy+y^2=31. \end{cases}$$

22. $$\begin{cases} x+y=5. \\ x^5+y^5=275. \end{cases}$$

23. $$\begin{cases} x^2+y^2=5. \\ x^6+y^6=65. \end{cases}$$

24. $$\begin{cases} x+2y=11. \\ x^2=61-4y^2. \end{cases}$$

25. $$\begin{cases} x^2y=84+xy^2. \\ x^3-y^3=316. \end{cases}$$

26. $$\begin{cases} 2xy=2a^{\frac{1}{2}}b^{\frac{1}{2}}. \\ x^4+y^4=a^2+b^2. \end{cases}$$

27. $$\begin{cases} x-y=\sqrt{x}+\sqrt{y}. \\ xy=4. \end{cases}$$

28. $$\begin{cases} x^3+y^3=\tfrac{7}{16}(x+y)^3. \\ xy=12. \end{cases}$$

29. $$\begin{cases} x+y+\sqrt{x+y}=12. \\ x-y+\sqrt{x-y}=2. \end{cases}$$

30. $$\begin{cases} x+y=6. \\ x^4+y^4=272. \end{cases}$$

31. $$\begin{cases} x^2-y^2=21. \\ x^3y+xy^3=290. \end{cases}$$

32. $$\begin{cases} x^2y+x=14. \\ x^4y^2+x^2=148. \end{cases}$$

33. $$\begin{cases} \dfrac{x}{y}+\dfrac{6\sqrt{x}}{\sqrt{y}}=16. \\ x+y=6. \end{cases}$$

34. $$\begin{cases} x^2y+xy^2=7. \\ x^4y^2+x^2y^4=25. \end{cases}$$

35. $$\begin{cases} x^3y+x=2. \\ x^6y^2+x^2=10. \end{cases}$$

36. $$\begin{cases} xy=6. \\ x^4+y^4=97. \end{cases}$$

37. $$\begin{cases} x^{-1}+y^{-1}=\tfrac{5}{6}. \\ x^{-2}+y^{-2}=\tfrac{13}{36}. \end{cases}$$

38. $$\begin{cases} x^2+y^3=28. \\ x^4+y^6=730. \end{cases}$$

39. $$\begin{cases} x^2y+xy^2=20. \\ x^3+y^3=65. \end{cases}$$

40. $$\begin{cases} x^{\frac{1}{4}}+y^{\frac{1}{5}}=6. \\ x^{\frac{3}{4}}+y^{\frac{3}{5}}=126. \end{cases}$$

41. $$\begin{cases} 2x+y=23-5\sqrt{2x+y}+1. \\ xy=4. \end{cases}$$

PROBLEMS.

236. 1. The sum of two numbers is 12, and their product is 20; what are the numbers?

2. The sum of two numbers multiplied by the less is 24, and their difference multiplied by their sum is 16; what are the numbers?

3. The sum of two numbers is 12, and the sum of their cubes is 468; what are the numbers?

4. The product of two numbers is six times their sum, and the sum of their squares is 325; what are the numbers?

5. The sum of two numbers is 5, and the sum of their fourth powers is 257; what are the numbers?

6. The difference of two numbers is 2, and the difference of their fifth powers is 242; what are the numbers?

7. The sum of two numbers is a, and the sum of their cube roots is $\sqrt[3]{a}$; what are the numbers?

8. The sum of the areas of two square fields is 2,000 square rods, and it takes 240 rods of fence to inclose them; what is the length of each side?

9. A has two cubical boxes: the difference of their contents is 342 cubic inches, and the difference of their edges is 6 inches; what is the length of each box?

10. Find a number containing two digits, such that the sum of the squares of the digits increased by one half their product equals the number; and if 18 be added to the number, the digits will be inverted.

11. A has three pieces of land: the first and second are unequal squares, and the third is as wide as the first and as long as the second. The area of the three pieces is 61 square rods, and three times the area of the smaller square increased by the area of the third is 68 square rods. Find the dimensions of each piece of land.

12. The fore wheel of a carriage makes 10 revolutions more than the hind wheel in going 120 yards; but if the circumference of each is increased 1 yard, the fore wheel will make only 6 more revolutions than the hind wheel in the same distance. Required the circumference of each wheel.

13. There are two cubical bins: the larger one contains 296 cubic inches more than the smaller one; and if 96 cubic inches be divided by the product of the edges, the quotient will be equal to the difference of the edges. What are the dimensions of the bins?

14. A regiment of 1,125 men was marching in two unequal squares; a detachment of 450 men was sent away, and the remainder was formed into a rectangular division whose front was equal to the side of the smaller square, and whose length was equal to the sum of the lengths of the two squares. Find the side of each square.

15. In a purse containing 15 coins each gold coin is worth as many dollars as there are silver coins, each silver coin is worth as many cents as there are gold coins, and the entire value of the coins is $50.50; how many coins are there of each sort?

16. A and B start at the same time from two different points, and travel toward each other; when they meet, it appears that A has traveled 18 miles more than B. It also appears that it would take A 4 days to travel the distance B has come, and B 9 days to travel the distance A has come. How far apart were they when they started?

17. A and B run a race around a three-mile course: in the first heat A reaches the winning post 3 minutes before B; in the second heat B increases his speed 3 miles per hour, and A diminishes his as much; and B then arrives at the winning post 3 minutes before A. Find at what rate each man ran in the first heat.

18. M and N run a race: M, who runs slower than N by 2 miles in 3 hours, starts first by 6 minutes, and they get to the 6-mile post together; what is the rate of each per hour?

THEORY OF QUADRATIC EQUATIONS.

237. Every quadratic equation may be reduced to the form of $x^2 + 2px = q$, in which p and q may be either positive or negative, integral or fractional.

PRINCIPLE I.

Every quadratic equation has two roots, and only two.

The general form of a quadratic is

$$x^2 + 2px = q$$

Writing the result (Art. 221), $x = -p + \sqrt{p^2 + q}$

$$x = -p - \sqrt{p^2 + q}$$

Which proves the principle.

PRINCIPLE II.

The sum of the two roots of a quadratic equation of the form $x^2 + 2px = q$ *is equal to the coefficient of* x *with its sign changed.*

Let r_1 and r_2 denote the roots of the quadratic equation $x^2 + 2px = q$; then, by Prin. I.,

$$\begin{array}{rl} r_1 = & -p + \sqrt{p^2 + q} \\ r_2 = & -p - \sqrt{p^2 + q} \\ \hline \end{array}$$

The *sum* $= r_1 + r_2 = -2p$

Which proves the principle.

PRINCIPLE III.

The product of the two roots of a quadratic equation of the form $x^2 + 2px = q$ *is equal to the known term with its sign changed.*

Let r_1 and r_2 denote the roots of the quadratic equation $x^2 + 2px = q$; then, by Prin. I.,

$$\begin{array}{rl} r_1 = & -p + \sqrt{p^2 + q} \\ r_2 = & -p - \sqrt{p^2 + q} \\ \hline & p^2 - p\sqrt{p^2 + q} \\ & \quad + p\sqrt{p^2 + q} - (p^2 + q) \\ \hline \end{array}$$

The *product* $= r_1 r_2 = p^2 - (p^2 + q) = -q$

Which proves the principle.

1. What are the sum and the product of the roots of the equation $x^2 + 6x = 27$?

By Prin. II., the sum of the roots is -6; and by Prin. III., the product of the roots is -27.

We can also determine the roots of the equation by knowing that their product is -27 and their sum -6. The only two numbers whose product is -27 and whose sum is -6 are $+9$ and -3. Hence, the roots of the equation $x^2 + 6x = 27$ are $+9$ and -3.

EXAMPLES.

Find by inspection the sum and the product of the roots of—

2. $x^2 + 8x = 20$.
3. $x^2 - 12x = 13$.
4. $x^2 + 14x = -40$.
5. $x^2 - 16x = -60$.
6. $x^2 + 10x = 11$.
7. $x^2 + 2ax = b$.
8. $x^2 - 20x = 21$.
9. $x^2 - mx = n$.
10. $8x^2 - 24x = 8$.
11. $3x^2 + 9x = -18$.

12. Form the equation whose roots are -7 and 4.

By Prin. II. and III., the equation is

$$x^2 - (-7 + 4)x = -4(-7)$$

Or,

$$x^2 + 3x = 28$$

Form the equations whose roots are—

13. 2 and 3.
14. 4 and -5.
15. -8 and 7.
16. -4 and -6.
17. a and b.
18. $a + b$ and $a - b$.
19. $a + \sqrt{b}$ and $a - \sqrt{b}$.
20. $m + \sqrt{-n}$ and $m - \sqrt{-n}$.
21. $-a$ and $-n$.
22. 5 and $-\frac{4}{5}$.

Find by inspection the roots of—

23. $x^2 + 18x = 19$.
24. $x^2 - 12x = 28$.
25. $x^2 + 2ax = m^2 - a^2$.
26. $x^2 - 8x = 33$.
27. $x^2 - 24x = 25$.
28. $x^2 - 10x = 56$.

29. $x^2 - 20x = 44.$

30. $x^2 - 2nx = m^2 - n^2.$

31. $x^2 - 16x = 105.$

32. $x^2 + 22x = -21.$

PRINCIPLE IV.

A quadratic equation of the form $x^2 + 2px = q$ *may be resolved into two binomial factors, of which the first term in each is* x *and the second term the roots with their signs changed.*

Let r_1 and r_2 be the roots of the equation $x^2 + 2px = q$; then, by Prin. II. and III., we have

$$x^2 - (r_1 + r_2)x = -r_1r_2$$

Or $$x^2 - (r_1 + r_2)x + r_1r_2 = 0$$

Factoring, $$(x - r_1)(x - r_2) = 0$$

Which proves the principle.

This principle is frequently used in solving equations of any degree.

1. Solve the equation $(x-4)(x-5)=0.$

Placing the factors separately equal to zero,

$$\left.\begin{array}{l} x - 4 = 0, \text{ or } x = 4 \\ x - 5 = 0, \text{ or } x = 5 \end{array}\right\} \textit{Ans.}$$

2. Solve the equation $(x-1)(x-2)(x-3)=0.$

Placing the factors separately equal to zero,

$$\left.\begin{array}{l} x - 1 = 0, \text{ or } x = 1 \\ x - 2 = 0, \text{ or } x = 2 \\ x - 3 = 0, \text{ or } x = 3 \end{array}\right\} \textit{Ans.}$$

3. Solve the equation $x^3 = 1.$

Transposing, $x^3 - 1 = 0$

Factoring, $(x - 1)(x^2 + x + 1) = 0$

Placing the factors separately equal to zero,

$$\left.\begin{array}{r} x - 1 = 0, \text{ or } x = 1, \\ x^2 + x + 1 = 0, \text{ or } x = \dfrac{-1 \pm \sqrt{-3}}{2} \end{array}\right\} \textit{Ans.}$$

4. Solve the equation $x^4 = 1$.

Transposing and factoring,

$$(x^2+1)(x+1)(x-1)=0$$

Placing the factors separately equal to zero,

$$\left.\begin{array}{l} x - 1 = 0, \text{ or } x = 1 \\ x + 1 = 0, \text{ or } x = -1 \\ x^2 + 1 = 0, \text{ or } x = \pm\sqrt{-1} \end{array}\right\} \textit{Ans.}$$

These examples illustrate that every equation has as many roots as there are units in its degree.

EXAMPLES.

Solve the following equations by factoring:

5. $(x-a)(x-b)=0$.
6. $x^3=8$.
7. $x^4=16$.
8. $x^6=1$.
9. $3x^2-x=0$.
10. $(x^2-9)(x^2-16)=0$.
11. $8x^3-32x=0$.
12. $x^2-10x=11$.
13. $x^3+x^2+x+1=0$.
14. $x^3+1=0$.
15. $x^2-8x=9$.
16. $x^3=64$.
17. $5x^3-125x=0$.
18. $(x-a)(x^2-2ax+b)=0$.
19. $x^3+3x^2-2x-6=0$.
20. $x^4-5x^3-x+5=0$.
21. $x^6-3x^4+3x^2-1=0$.
22. $x^4+x^2+1=0$.
23. $x^4+a^2x^2+a^4=0$.
24. $x^4=4$.
25. $(x+3)(x^2+7x+12)=0$.
26. $4x^2+16x+12=0$.
27. $x^5-x=0$.
28. $x^6-16x^2=0$.

In the general equation of the second degree, $x^2+2px=q$, p and q may be either positive or negative. This will give rise to **four distinct forms**, as follows:

(1.) $x^2+2px=q$.
(2.) $x^2-2px=q$.
(3.) $x^2+2px=-q$.
(4.) $x^2-2px=-q$.

the greater.

PRINCIPLE VII.

negative.

hence, both are +, or both −.

$2p$: hence, both roots are negative.

PRINCIPLE VIII.

both roots are posi

$-q$: hence,

$2p$: hence,

VALUES OF p

forms:

r_1

$r_2 =$